Stable Ranks for C*-algebras and Rokhlin Actions

(Based on joint work with Ms. Anshu)

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Lebesgue Covering Dimension
Let $X$ be a compact Hausdorff space and $\mathcal{V}$ be an open cover of $X$. We say that $\mathcal{V}$ has order $m$ if every point in $X$ belongs to at most $m$ elements of $\mathcal{V}$.

**Definition**

The (Lebesgue covering) dimension of $X$ is at most $n$ if every open cover of $X$ has a refinement $\mathcal{V}$, such that

$$\text{order}(\mathcal{V}) \leq n + 1$$
Figure 1: Dimension of a Circle (Source: Wikipedia)

The open cover (on the right) has a refinement (on the left) of order 2. Hence,

$$\text{dim}(S^1) \leq 1$$
If $C$ denotes the Cantor set, then

$C$ is totally disconnected

(ie. $C$ has a basis consisting of cl-open sets)

Every open cover of $C$ has a refinement consisting of disjoint sets

(ie. of order 1). Hence,

$$\dim(C) = 0$$
Theorem

Let $X$ be a compact Hausdorff space. Then $\dim(X)$ is the least integer $n$ such that, for any continuous function

$$f : X \rightarrow \mathbb{R}^{n+1}$$

and any $\epsilon > 0$, there is a continuous function

$$g : X \rightarrow \mathbb{R}^{n+1}$$

such that $\|f - g\|_\infty < \epsilon$ and

$$0 \notin g(X)$$

This says that 0 is an unstable value for $f$. 
**Theorem**

Let $X$ be a compact Hausdorff space. Then $\dim(X)$ is the least integer $n$ such that any $(n+1)$-tuple

$$(f_1, f_2, \ldots, f_{n+1}) \in C(X, \mathbb{R})^{n+1}$$

can be approximated arbitrarily closely by a tuple

$$(g_1, g_2, \ldots, g_{n+1}) \in C(X, \mathbb{R})^{n+1}$$

such that

$$\sum_{i=1}^{n+1} g_i^2$$

is a strictly positive function.
Stable Ranks for C*-algebras
Connection to C*-algebras

**Definition**

A C*-algebra is a Banach algebra $A$ together with an involution $a \mapsto a^*$ satisfying certain conditions.

Example:

- $\mathbb{C}$ is a C*-algebra.
- If $X$ is a compact Hausdorff space, $C(X) = C(X, \mathbb{C})$ is a C*-algebra.
- If $H$ is a Hilbert space, $B(H)$ is a C*-algebra.
Stable Rank for C*-algebras

**Definition (Rieffel (1982))**

Let $A$ be a C*-algebra. The *topological stable rank* (tsr) of $A$ is the least integer $n$ such that any tuple

$$(a_1, a_2, \ldots, a_n) \in A^n$$

can be approximated arbitrarily closely by a tuple

$$(b_1, b_2, \ldots, b_n) \in A^n$$

such that

$$\sum_{i=1}^{n} b_i^* b_i$$

is invertible in $A$.

If no such integer exists, we write $\text{tsr}(A) = +\infty$. 
Examples

- Since a complex valued function can be thought of as a pair of real-valued functions, we have

\[ tsr(C(X)) = \left\lceil \frac{\dim(X)}{2} \right\rceil + 1 \]

where \( \lceil x \rceil \) is the ‘integer part’ of \( x \).

- \( tsr(\mathbb{C}) = 1 \)

- More generally, if \( A \) is a finite dimensional C*-algebra, then

\[ tsr(A) = 1 \]
Examples

- For any C*-algebra $A$,

$$tsr(A) = 1 \iff GL(A) \text{ is dense in } A$$

- If $S \in B(\ell^2)$ denotes the right-shift operator, then $S$ cannot be approximated by invertibles. Hence, if $A := C^*(S)$, then

$$tsr(A) \neq 1$$

In fact, $tsr(A) = 2$.

- If $H$ is an infinite dimensional Hilbert space, then

$$tsr(B(H)) = +\infty$$
How are stable ranks useful?

- Knowing the stable rank of an algebra helps answer questions in ‘nonstable’ K-theory. ie. One can use K-theoretic information to extract information about elements (projections or unitaries) in the algebra.
- Algebras with stable rank 1 have nice regularity properties that are useful in classification.
Group Actions on C*-algebras
Standing assumption:

- $G$ = Finite group
- $A$ = Unital, separable C*-algebra.

Define

$$\text{Aut}(A) := \{\text{involution-preserving automorphisms of } A\}$$

Note that Aut(A) is a group under composition.

**Definition**

A *group action* of $G$ on $A$ is a group homomorphism

$$\alpha : G \rightarrow \text{Aut}(A)$$
• If $G \curvearrowright X$ is an action of $G$ on a compact Hausdorff space $X$, then it induces an action $\alpha : G \curvearrowright C(X)$ by

$$\alpha_g(f)(x) := f(g^{-1} \cdot x)$$

• Furthermore, every $\text{C}^*$-algebra action of $G$ on $C(X)$ arises in this way.

• If $\alpha \in \text{Aut}(A)$ is an automorphism of order $N$, then it induces a group action

$$\alpha : \mathbb{Z}_N \rightarrow \text{Aut}(A)$$
Given an action $G \curvearrowright X$ on a space $X$, one can take the quotient

$$X/G$$
Crossed Product C*-algebras

Given an action $G \curvearrowright X$ on a space $X$, one can take the quotient $X/G$.

If $\alpha : G \curvearrowright A$ is an action of $G$ on a C*-algebra $A$, then the analogous object to consider is $A \rtimes_\alpha G$.

The crossed product.

One can think of it like a semi-direct product of $G$ with $A$. 
Example

If $\alpha \in \text{Aut}(A)$ has order $N$, then the crossed product

$$A \rtimes_{\alpha} \mathbb{Z}_N$$

is the subalgebra of $M_N(A)$ generated by elements of the form

$$\pi(a) := \begin{pmatrix}
  a \\
  \alpha(a) \\
  \vdots \\
  \alpha^{N-1}(a)
\end{pmatrix}$$

and $U := \begin{pmatrix}
  0 & 1 \\
  0 & 1 \\
  \vdots & \vdots \\
  0 & 1
\end{pmatrix}$

Once can check that

$$\pi(\alpha(a)) = U\pi(a)U^{-1}$$
The Main Question

**Question**

If $\alpha : G \to \text{Aut}(A)$ is a group action of a finite group on a C*-algebra, can we estimate $\text{tsr}(A \rtimes_{\alpha} G)$ in terms of $\text{tsr}(A)$?

In 2007, Jeong, Osaka, Phillips and Teruya proved that

$$\text{tsr}(A \rtimes_{\alpha} G) \leq \text{tsr}(A) + |G| - 1$$
The Main Question

We had two objectives:

- Can we improve this estimate if we impose certain conditions on the action?
- Can we also find estimates for other ‘ranks’? (Topological stable rank is not the only dimension theory for C*-algebras)
Rokhlin Actions
An action $\alpha : G \curvearrowright A$ is said to have the **Rokhlin property** if, for any finite set $F \subset A$ and any $\epsilon > 0$, there are projections $\{e_g : g \in G\} \subset A$ such that

1. $\sum_{g \in G} e_g = 1_A$ [Partition of Unity]
2. $\alpha_g(e_h) = e_{gh}$ [Permuted by $G$]
3. $\|e_g a - ae_g\| < \epsilon$ [Approximately commutes with $F$]
Example: Commutative C*-algebras

If $A = C(X)$, then a projection in $A$ corresponds to a cl-open set in $X$. Hence, the condition above implies that

$X$ is totally disconnected.

(ie. $X$ has a basis consisting of cl-open sets)

Furthermore, if $X$ is totally disconnected, the Rokhlin property is equivalent to saying that

the action is free.
If $A$ is non-commutative, then $A$ tends to admit a Rokhlin action if $A$ is low-dimensional.

(ie. $A$ has a lot of projections)
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(ie. $A$ has a lot of projections)

Not every C*-algebra admits a Rokhlin action of a finite group (for instance, $A_{\theta}$ and $O_{\infty}$ do not).
Example: $A = M_{2\infty}$

Let $A$ be the UHF algebra of type $2^\infty$

$$A = \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$$

Let $v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{C})$ and

$$\text{Ad}(v) : M_2(\mathbb{C}) \to M_2(\mathbb{C})$$

be given by $a \mapsto vav^*$

Then

$$\alpha := \bigotimes_{n=1}^{\infty} \text{Ad}(v)$$

is an order 2 automorphism of $A$, so defines an action

$$\alpha : \mathbb{Z}_2 \curvearrowright M_{2\infty}$$

This action has the Rokhlin property.
Main Result

Theorem (Anshu, PV (2020))

If \( \alpha : G \to \text{Aut}(A) \) is an action of a finite group on a unital, separable C*-algebra with the Rokhlin property, then

\[
\text{tsr}(A \rtimes_\alpha G) \leq \left\lceil \frac{\text{tsr}(A) - 1}{|G|} \right\rceil + 1
\]
• Analogous inequalities also hold for a number of other ‘ranks’ for C*-algebras, including:
  1. Connected stable rank
  2. General stable rank
  3. Real Rank
• In 2012, Osaka and Phillips proved that, if the action is Rokhlin and $\text{tsr}(A) = 1$, then

$$\text{tsr}(A \rtimes_{\alpha} G) = 1$$

So our theorem is a strengthening of that result.
Thank you!