KOROVKIN’S THEOREM

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Abstract. We discuss an approximation theorem due to Korovkin, which provides a short proof of Weierstrass’ approximation theorem. We then discuss how this relates to the Choquet boundary of the corresponding function system.

1. Korovkin’s Theorem

Definition 1.1.
(1) Let \(X\) be a compact Hausdorff space, and write
\[
C(X) := \{f : X \to \mathbb{C} \text{ continuous}\}
\]
equipped with the sup-norm.
(2) For \(f \in C(X)\), we write \(f \geq 0\) if \(f(x) \in [0, \infty)\) for all \(x \in X\).
(3) Write \(f \geq g\) if \(f(x), g(x) \in \mathbb{R}\) and \(f(x) \geq g(x)\) for all \(x \in X\).
(4) A linear map \(T : C(X) \to C(X)\) is said to be positive if
\[
f \geq 0 \Rightarrow T(f) \geq 0
\]
Equivalently, \(T\) is order preserving, i.e. \(f \leq g \Rightarrow T(f) \leq T(g)\).
(5) Write \(1\) for the constant function 1.

Theorem 1.2 (Korovkin (1953)). Let \(T_n : C[0, 1] \to C[0, 1]\) be a sequence of positive operators such that
\[
T_n(g) \to g \text{ for } g \in \{1, e_1, e_2\}
\]
where \(e_i(t) := t^i\). Then
\[
T_n(f) \to f \quad \forall f \in C[0, 1]
\]

Example 1.3 (Weierstrass’ Approximation Theorem). Define \(B_n : C[0, 1] \to C[0, 1]\) by
\[
B_n(f)(x) := \sum_{k=0}^{n} f \left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}
\]
Then \(B_n\) is linear, and positive. Also,
\[
B_n(1)(x) = (x + (1-x))^n = 1(x)
\]
\[
B_n(e_1)(x) = e_1(x)
\]
\[
B_n(e_2)(x) = \frac{n-1}{n} e_2(x) + \frac{1}{n} e_1(x)
\]
Hence, \(B_n(g) \to g\) for \(g \in \{1, e_1, e_2\}\), so by Korovkin’s theorem,
\[
B_n(f) \to f \quad \forall f \in C[0, 1]
\]

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2. THE CHOQUET BOUNDARY

Definition 2.1. A subset $G \subset C(X)$ is called a Korovkin set if, for any sequence of positive operators $T_n : C(X) \to C(X)$,

$$T_n(g) \to g \quad \forall g \in G \Rightarrow T_n(f) \to f \quad \forall f \in C(X)?$$

Question: Given $G \subset C(X)$, how do we determine if it is a Korovkin set?

Definition 2.2. Let $A \subset C(X)$ be a linear space, and $\mu$ a (Borel, regular) probability measure on $X$. We say that $\mu$ represents the point $x \in X$ with respect to $A$ if

$$f(x) = \int_X f d\mu \quad \forall f \in A$$

We write Rep$(x, A)$ for the set of all such measures.

Example 2.3.

(1) If $x \in X$, and $\delta_x$ denotes the delta measure supported at $x$, then $\delta_x \in$ Rep$(A, x)$ for any $A$.

(2) If $X = [0, 1]$ and $A \subset C[0, 1]$ denotes the subset of affine maps ($f$ is affine if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ for all $t \in [0, 1]$ and $x, y \in [0, 1]$). Then

$$\frac{\delta_0 + \delta_1}{2} \in$ Rep$(A, 1/2)$

(3) If $A \subset C(X)$ is dense in $C(X)$ and $\mu \in$ Rep$(A, x)$, then

$$\int f d\delta_x = \int f d\mu \quad \forall f \in A$$

So by continuity, it must follow that $\mu = \delta_x$. Hence,

Rep$(A, x) = \{\delta_x\} \quad \forall x \in X$

(4) Let $D \subset \mathbb{C}$ denote the open unit disc, $X := \overline{D}$ and $A \subset C(X)$ the set of functions that are analytic on $D$. This is called the disc algebra. If $f \in A$ and $z := re^{it} \in D$, then $0 \leq r < 1$, and so the Poisson integral formula holds

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})P_r(\theta, t)d\theta$$

This gives a measure $\mu_{r,t} \in$ Rep$(A, z)$. Note that $\mu_{r,t} \neq \delta_z$ because $\mu_{r,t}$ is supported on $T$, the unit circle.

Definition 2.4. Let $A \subset C(X)$ be a linear subspace. A point $x \in X$ is called a boundary point with respect to $A$ if $\delta_x$ is the only probability measure that represents $x$ with respect to $A$. We write

$$\partial A(X) = \{x \in X : \text{Rep}(A, x) = \{\delta_x\}\}$$

for the set of boundary points with respect to $A$. This is called the Choquet boundary of $A$.

Note: It is not obvious that $\partial A(X) \neq \emptyset$, but it is non-empty if $A$ separates points of $X$.

Example 2.5.

(1) If $A \subset C(X)$ is dense, then $\partial A(X) = X$

(2) If $A \subset C[0,1]$ is the set of affine functions, then $1/2 \notin \partial A(X)$

(3) If $A \subset C(\mathbb{D})$ denotes the disc algebra, then the Poisson integral formula shows that, if $z \in \mathbb{D}$, then $z \notin \partial A(X)$. Hence, $\partial A(X) \subset T$, the unit circle. In fact, one can show that $\partial A(X) = T$. 

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Let \( G \subset C(X) \) containing \( 1 \), and define
\[
A_G := \text{span}(G \cup G^*)
\]
where \( G^* := \{ \overline{f} : f \in G \} \). This is called the function system generated by \( G \).

**Theorem 2.6** (Saskin (1967), Davidson-Kennedy (2016)). If \( G \subset C(X) \) be a subset containing \( 1 \) that separates points of \( X \), then \( G \) is a Korovkin set if and only if \( \partial_{A_G}(X) = X \)

This is proved in the metrizable setting by Saskin (See [Berens and Lorentz, Theorems 1 and 3]). The non-metrizable case (and much more) is due to Davidson-Kennedy [Davidson and Kennedy, Theorem 7.5].

**Proof.** Suppose \( X \) is metrizable.

1. **Necessity:** Let \( G \subset C(X) \), and \( x_0 \in X \). Suppose \( \mu \neq \delta_{x_0} \) is a measure that represents \( x_0 \) with respect to \( A \), then we show that \( A \) is not a Korovkin set. Define
\[
G_n := \{ x \in X : d(x, x_0) \geq \frac{1}{n} \}
\]
Then \( G_n \) is closed and \( x_0 \notin G_n \), so by Urysohn’s lemma, \( \exists \varphi_n \in C(X) \) such that \( 0 \leq \varphi_n \leq 1 \),
\[
\varphi_n(x_0) = 1 \text{ and } \varphi_n|_{G_n} = 0
\]
Define \( T_n : C(X) \to C(X) \) by
\[
T_n(f) := \mu_0(f)\varphi_n + f \cdot [1 - \varphi_n]
\]
Then \( T_n \) is positive linear, and if \( g \in A \), then
\[
T_n(g) = g(x_0)\varphi_n + g \cdot [1 - \varphi_n] \to g
\]
But there exists \( f \in C(X) \) such that \( \mu_0(f) \neq f(x_0) \), so
\[
T_n(f)(x_0) = \mu_0(f) \quad \forall n \in \mathbb{N}
\]
Hence, \( T_n(f) \not\to f \)

2. **Sufficiency:** Suppose \( G \) has the property that \( \partial_{A_G}(X) = X \), and \( \{T_n\} \) is a sequence of positive linear operators such that \( T_n(g) \to g \) for all \( g \in G \). We show that \( T_n(f) \to f \) for all \( f \in C(X) \).

Suppose not, then there exists \( f_0 \in C(X) \), and \( \epsilon > 0 \) and a sequence \( \{n_k\} \) of natural numbers, and points \( x_k \in X \) such that
\[
|T_{n_k}(f_0)(x_k) - f_0(x_k)| \geq \epsilon
\]
By compactness, we may assume that \( x_k \to x_0 \). Define measures \( \mu_k \) by
\[
\mu_k(f) := T_{n_k}(f)(x_k)
\]
Observe that, for any possible positive operator \( T \),
\[
|T(f)| \leq T(|f|) \leq \|f\|T(1)
\]
Hence, \( \|T\| \leq T(1) \). Since
\[
\|T_n\| = \|T_n(1)\| \to \|1\| = 1
\]
we may normalize to assume that \( \|T_n\| = 1 \) for all \( n \in \mathbb{N} \), whence each \( \mu_k \) is a probability measure. Thus, \( \{\mu_k\} \) is a subset of the closed unit ball of \( C(X)' \), the dual of \( C(X) \), and so must have a convergent subsequence by Banach-Alaoglu. Once again, we may assume without loss of generality that \( \mu_k \to \mu \) in the weak-
Note that \( \mu_k(g) \to g(x_0) \) for all \( g \in G \), so

\[
\mu(g) = g(x_0) = \delta_{x_0}(g) \quad \forall g \in G
\]

Since \( \partial_{A_G}(X) = X \), it follows that \( \mu = \delta_{x_0} \). In particular,

\[
T_{n_k}(f_0)(x_k) = \mu_k(f_0) \to f_0(x_0)
\]

This contradicts Equation 1.

\[\square\]

**Corollary 2.7.** Let \( G \subset C(X) \) be a subset that contains 1 and has the property that, for any two distinct points \( x, y \in X \), \( \exists g \in A_G \) positive such that

\[
0 = g(x) < g(y)
\]

Then \( G \) is a Korovkin set.

**Proof.** Let \( x \in X \) and let \( \mu \) be a probability measure on \( X \) such that \( \mu \neq \delta_x \). Then \( \exists y \in X \) such that \( y \neq x \) and \( y \) is in the support of \( \mu \). Choose \( g \in A_G \) as above, then

\[
\int_X gd\mu > 0 = g(x)
\]

So \( \mu \) does not represent \( x \) with respect to \( A_G \). Hence, \( x \in \partial_X(A_G) \). Thus,

\[
X = \partial_{A_G}(X)
\]

so Saskin’s theorem applies. \( \square \)

**Example 2.8.**

1. If \( X = [a,b] \) and \( 0 \leq s < t \), then \( G := \{1, x^s, x^t\} \) is a Korovkin set for \( C[a,b] \), because if \( c \in [a,b] \), then define

\[
f(x) := t - s - t \left( \frac{x}{c} \right)^s + s \left( \frac{x}{c} \right)^t
\]

Then \( f \in A_G \) and \( f(c) = 0 = f'(c) \). Also, \( f'(x) \) changes sign only at \( x = c \), so the above Corollary applies.

2. If \( X \subset \mathbb{R}^n \) is compact and we consider \( C_\mathbb{R}(X) \), the space of real-valued continuous functions on \( X \), then the above theorem also holds in that setting. Furthermore, if \( \{f_1, f_2, \ldots, f_k\} \subset C_\mathbb{R}(X) \) separates points, then set

\[
G := \{1, f_1, f_2, \ldots, f_k, \sum_{i=1}^k f_i^2\}
\]

(For instance, we may take \( f_i := \pi_i \), then \( i^{th} \) coordinate projection). Again \( G \) is a Korovkin set because if \( x \in X \), then

\[
g(y) := \sum_{i=1}^k (f_i(y) - f_i(x))^2 \in A_G
\]

and \( 0 = g(x) < g(y) \) for any \( y \neq x \).
3. A generalization: Strong Korovkin sets

Let $X$ be a compact Hausdorff space.

**Definition 3.1.** A representation of $C(X)$ is an algebra homomorphism $\pi : C(X) \to \mathcal{B}(H)$ such that $\pi(f^*) = \varphi(f)^*$. A set $G \subset C(X)$ is said to be a strong Korovkin set if, for any representation $\pi : C(X) \to \mathcal{B}(H)$ and any sequence of positive linear maps $\varphi_n : C(X) \to \mathcal{B}(H)$

$$\|\varphi_n(g) - \pi(g)\| \to \forall g \in G \Rightarrow \|\varphi_n(f) - \pi(f)\| \to 0 \ \forall f \in C(X)$$

**Example 3.2.** Every strong Korovkin set is a Korovkin set, because we may embed $C(X) \to \mathcal{B}(H)$ for some Hilbert space $H$, and consider $\pi : C(X) \to \mathcal{B}(H)$ to be the inclusion map.

What is remarkable is that the two notions are actually the same.

**Theorem 3.3** (Davidson-Kennedy (2016)). A subset $G \subset C(X)$ is a strong Korovkin set if and only if it is a Korovkin set.

**Definition 3.4.** Let $C$ be a C*-algebra and $G \subset C$ a set. $G$ is called a strong Korovkin set if, for any non-degenerate representation $\pi : C \to \mathcal{B}(H)$ of $C$, and any sequence of completely positive linear maps $\varphi_n : C \to \mathcal{B}(H)$,

$$\|\varphi_n(g) - \pi(g)\| \to \forall g \in G \Rightarrow \|\varphi_n(f) - \pi(f)\| \to 0 \ \forall f \in C$$

If this happens, the operator system $A_G$ is called hyperrigid.

**Example 3.5** (Arveson (2011)).

1. Let $X \in \mathcal{B}(H)$ be a self-adjoint operator and $A = C^*(X)$, then $G = \{X, X^2\}$ is a strong Korovkin set.
2. If $A = \mathcal{O}_n$ is the Cuntz algebra, then the standard generators $G = \{v_1, v_2, \ldots, v_n\}$ is a strong Korovkin set.

The analog of Saskin’s theorem (Theorem 2.6) in this context is an open question, called Arveson’s Hyperrigidity Conjecture [Arveson].

**References**


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