Connected Stable Ranks of C*-algebras

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The Connected Stable Rank

Pullbacks of C*-algebras

Tensor products by commutative C*-algebras
The Connected Stable Rank
Let $A$ be a unital $C^*$-algebra. Fix $n \in \mathbb{N}$ and write $\pi_0(GL_n(A))$ for the set of path components of $GL_n(A)$. Define

$$\theta_A : GL_n(A) \to GL_{n+1}(A) \text{ given by } a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

**Definition**

$K_1(A)$ is defined as the inductive limit

$$\ldots \to \pi_0(GL_n(A)) \xrightarrow{\theta_A} \pi_0(GL_{n+1}(A)) \to \ldots$$

This is a group under the operation

$$[u] + [v] := [u \oplus v] = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$
Let $u, v, w \in GL_n(A)$. Write

$$u \sim_h v$$

if they are path connected in $GL_n(A)$.

When is it true that

$$u \oplus w \sim_h v \oplus w \Rightarrow u \sim_h v$$

This is equivalent to asking

$$[u] = [v] \text{ in } K_1(A) \Rightarrow u \sim_h v$$
Fix $n \in \mathbb{N}$. A vector $\underline{a} = (a_1, a_2, \ldots, a_n) \in A^n$ is called unimodular if

$$Aa_1 + Aa_2 + \ldots Aa_n = A$$

Write $Lg_n(A)$ for the set of unimodular vectors.
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Write $Lg_n(A)$ for the set of unimodular vectors.

Note that $GL_n(A)$ acts on $Lg_n(A)$ by left multiplication.

$$T \cdot \underline{a} := T(a^t)$$

Write

$$GL_n^0(A) = \{ T \in GL_n(A) : T \sim h I \}$$

for the connected component of $1_{A^n} \in GL_n(A)$. Note that this is a subgroup of $GL_n(A)$, and hence acts on $Lg_n(A)$ as well.
Definition (Rieffel, 1982)

The connected stable rank of $A$, denoted by $csr(A)$, is

\[
\min\{n \in \mathbb{N} : GL^0_m(A) \text{ acts transitively on } Lg_m(A) \ \forall m \geq n\}
\]

\[
= \min\{n \in \mathbb{N} : Lg_m(A) \text{ is connected for all } m \geq n\}
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**Definition (Rieffel, 1982)**

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Write $C(\mathbb{T}, A) := \{f : \mathbb{T} \to A : f \text{ is continuous}\}$.

**Theorem (Rieffel, 1982)**

Let $n \geq csr(C(\mathbb{T}, A)) - 1$, and $u, v \in GL_n(A)$. Then

$$[u] = [v] \text{ in } K_1(A) \Rightarrow u \sim_h v$$

In fact,

$$K_1(A) \cong \pi_0(GL_n(A))$$
Homotopy Invariance

**Theorem (Nistor, 1987)**

If $A$ and $B$ are homotopy equivalent as C*-algebras, then

$$csr(A) = csr(B)$$

This makes it both interesting, and (sometimes) hard to compute.

**Example**

If $X$ is contractible, then

$$csr(C(X)) = 1$$

regardless of the covering dimension of $X$. 
Known Facts

- Given a short exact sequence $0 \to J \to A \to B \to 0$,
  \[
  \text{csr}(A) \leq \max\{\text{csr}(J), \text{csr}(B)\}
  \]
- If $\pi : A \to B$ is a split epimorphism, then
  \[
  \text{csr}(B) \leq \text{csr}(A)
  \]
- If $\pi$ does not split, then there is no relation between $\text{csr}(B)$ and $\text{csr}(A)$.

Example

\[
C(\mathbb{D}^n) \to C(\mathbb{S}^{n-1})
\]

but $\text{csr}(C(\mathbb{D}^n)) = 1$ and $\text{csr}(C(\mathbb{S}^{n-1})) \neq 1$ for $n > 5$. 
Pullbacks of C*-algebras
Given two \(*\)-homomorphisms \(\delta : B \rightarrow D, \gamma : C \rightarrow D\), we define the pullback

\[ A = \{(b, c) \in B \oplus D : \delta(b) = \gamma(c)\} \]

This is described by a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow{\beta} & & \downarrow{\delta} \\
C & \xrightarrow{\gamma} & D
\end{array}
\]

where \(\alpha(b, c) = b\), and \(\beta(b, c) = c\).
Theorem

Given a pullback

\[ A \longrightarrow B \]
\[ \downarrow \gamma \downarrow \delta \]
\[ \downarrow \] \[ C \longrightarrow D \]

where either \( \gamma \) or \( \delta \) is surjective, then

\[
csr(A) \leq \max\{csr(B), csr(C), csr(C(\mathbb{T}, D))\}
\]
Example: CW-complexes

If $X = X_0 \cup \varphi \mathbb{D}^n$ is a CW-complex where $\varphi : \mathbb{S}^{n-1} \to X_0$ is the attaching map, then $C(X)$ is a pullback

\[
\begin{array}{c}
C(X) \longrightarrow C(X_0) \\
\downarrow \quad \downarrow \\
C(\mathbb{D}^n) \quad \gamma \quad C(\mathbb{S}^{n-1})
\end{array}
\]

where $X_0$ is a CW-complex, and $\gamma$ is the restriction map. Thus, $C(X)$ is an iterated pullback.
A non-commutative CW-complex is defined inductively by

1. $A_0$ is finite dimensional.

2. Having defined $\{A_0, A_1, \ldots, A_k\}$, we define

$$
\begin{array}{c}
A_{k+1} \\
\downarrow \\
C(\mathbb{D}^n) \otimes F_k
\end{array} \xrightarrow{\gamma} \begin{array}{c}
A_k \\
\downarrow \\
C(\mathbb{S}^{n-1}) \otimes F_k
\end{array}
$$

where $F_k$ is finite dimensional, and $\gamma$ is the restriction map.

The topological dimension of $A_k$ is defined to be $k$.

If $A_0 = F_k = \mathbb{C}$, this defines a CW-complex.
Theorem

Let $A$ be a NCCW complex of topological dimension $n$, then

$$csr(A) \leq \left\lceil \frac{n}{2} \right\rceil + 1$$

In particular, if $X$ is a CW-complex of dimension $n$, then

$$csr(C(X)) \leq \left\lceil \frac{n}{2} \right\rceil + 1$$
Tensor products by commutative C*-algebras
**Theorem**

Let $A$ be a unital C*-algebra, and $X$ be a compact Hausdorff space of dimension $n$, then

$$csr(C(X) \otimes A) \leq \max\{csr(C(\mathbb{T}^k) \otimes A) : 0 \leq k \leq n\}$$

It is known that

$$csr(C(\mathbb{T}^k)) = \left\lceil \frac{k}{2} \right\rceil + 1$$

**Corollary (Nistor, 1987)**

If $X$ is a compact Hausdorff space of dimension $n$

$$csr(C(X)) \leq \left\lceil \frac{n}{2} \right\rceil + 1$$
**K-stable C*-algebras**

**Definition (Thomsen, 1991)**

A C*-algebra $A$ is said to be $K$-stable if

$$\theta_A : GL_{k-1}(A) \to GL_k(A)$$

is a weak homotopy equivalence for all $k \geq 2$.

Examples include the Irrational Rotation Algebra $A_\theta$ ($\theta \notin \mathbb{Q}$), the Cuntz Algebra $\mathcal{O}_n$, the Jiang Su Algebra $\mathcal{Z}$.

**Theorem**

Let $X$ be a compact Hausdorff space and $A$ be $K$-stable, then

$$csr(C(X) \otimes A) = \begin{cases} 
csr(A) & : \text{if } csr(A) \geq 2 \\
1 \text{ or } 2 & : \text{if } csr(A) = 1 
\end{cases}$$
Given a C*-algebra $A$, we may also define the *general stable rank* of $A$, denoted $gsr(A)$.

- $gsr(\cdot)$ is also homotopy invariant.
- It helps understand certain non-stable behaviour in the $K_0$-group of $A$. 

Both these invariants have been studied in the following article: *Homotopical Stable Ranks for certain C*-algebras* (Studia Mathematica, 2018). Thank you!
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Thank you!