Homotopical Stable Ranks of C*-algebras

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Outline

Non-Stable $K$-theory

Computing these stable ranks

Examples
Non-Stable $K$-theory
Let $A$ be a unital C*-algebra. Define

$$\mathcal{P}(A) := \text{isomorphism classes of f.g. projective modules over } A$$

Write $[P]$ for the class of a projective module, and define

$$[P] + [Q] := [P \oplus Q]$$

Then $(\mathcal{P}(A), +)$ is an abelian semigroup with identity $[0]$.

**Definition**

$K_0(A)$ is the Grothendieck completion of $(\mathcal{P}(A), +)$. 
Let $A$ be a $C^*$-algebra, $P, Q, R$ projective modules over $A$.

When is it true that

$$P \oplus R \cong Q \oplus R \Rightarrow P \cong Q?$$

This is equivalent to asking

$$[P] = [Q] \text{ in } K_0(A) \Rightarrow P \cong Q?$$
Cancellation of Projections

Let $A$ be a $C^*$-algebra, $P$, $Q$, $R$ projective modules over $A$.

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This is equivalent to asking

$$[P] = [Q] \text{ in } K_0(A) \Rightarrow P \cong Q?$$

If $A = C(X)$, then $A$ satisfies cancellation of projections if

$$\dim(X) \leq 1$$

The appropriate dimension theory for $C^*$-algebras was introduced by Rieffel in the 1980s, and is called topological stable rank.
We may also ask some weaker questions:

- When is it true that

\[ P \oplus A \cong A^m \Rightarrow P \cong A^{m-1}? \]

Answer: If \( A = C(X) \), then this happens if \( X \) is contractible.

- Does there exists \( n \geq 1 \) such that, for all \( m \geq n \),

\[ P \oplus A \cong A^m \Rightarrow P \cong A^{m-1}? \]

**Definition (Rieffel, 1982)**

The least \( n \geq 1 \) satisfying the above property is called the **general stable rank** of \( A \), denoted by \( gsr(A) \).
The $K_1$ group

Let $A$ be a unital C*-algebra. Fix $n \in \mathbb{N}$ and write $\pi_0(GL_n(A))$ for the set of path components of $GL_n(A)$. Define

$$\theta_A : GL_n(A) \to GL_{n+1}(A) \text{ given by } a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

**Definition**

$K_1(A)$ is defined as the inductive limit

$$\ldots \to \pi_0(GL_n(A)) \xrightarrow{\theta_A} \pi_0(GL_{n+1}(A)) \to \ldots$$

This is a group under the operation

$$[u] + [v] := [u \oplus v] = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$
Cancellation of Invertibles

Let $u, v, w \in GL_n(A)$. Write

$$u \sim_h v$$

if they are path connected in $GL_n(A)$.

When is it true that

$$u \oplus w \sim_h v \oplus w \Rightarrow u \sim_h v$$

This is equivalent to asking

$$[u] = [v] \text{ in } K_1(A) \Rightarrow u \sim_h v$$
Fix $n \in \mathbb{N}$. A vector $a = (a_1, a_2, \ldots, a_n) \in A^n$ is called unimodular if

$$Aa_1 + Aa_2 + \ldots Aa_n = A$$

Write $Lg_n(A)$ for the set of unimodular vectors.
Fix $n \in \mathbb{N}$. A vector $a = (a_1, a_2, \ldots, a_n) \in A^n$ is called unimodular if

$$Aa_1 + Aa_2 + \ldots + Aa_n = A$$

Write $Lg_n(A)$ for the set of unimodular vectors.

Note that $GL_n(A)$ acts on $Lg_n(A)$ by left multiplication.

$$T \cdot a := T(a^t)$$

**Equivalent Definition (Rieffel, 1982)**

The least $n \geq 1$ such that $GL_n(A)$ acts transitively on $Lg_n(A)$ is the general stable rank of $A$. 
Write $GL_0^n(A)$ for the connected component of $1_{A^n} \in GL_n(A)$. Note that this is a subgroup of $GL_n(A)$, and hence acts on $Lg_n(A)$ as well.

**Definition (Rieffel, 1982)**

The least $n \geq 1$ such that $GL_0^n(A)$ acts transitively on $Lg_n(A)$ is called the **connected stable rank** of $A$, denoted by $csr(A)$.
Write \( GL_n^0(A) \) for the connected component of \( 1_{A^n} \in GL_n(A) \). Note that this is a subgroup of \( GL_n(A) \), and hence acts on \( Lg_n(A) \) as well.

**Definition (Rieffel, 1982)**

The least \( n \geq 1 \) such that \( GL_n^0(A) \) acts transitively on \( Lg_n(A) \) is called the **connected stable rank** of \( A \), denoted by \( csr(A) \).

Write \( C(\mathbb{T}, A) := \{ f : \mathbb{T} \to A : f \text{ is continuous} \} \).

**Theorem (Rieffel, 1982)**

If \( n \geq csr(C(\mathbb{T}, A)) - 1 \), and \( u, v \in GL_n(A) \) such that \( [u] = [v] \) in \( K_1(A) \), then

\[
    u \sim_h v \text{ in } GL_n(A)
\]
Computing these stable ranks
Homotopy Invariance

**Theorem (Nistor (1987), Nica (2011))**

If $A$ and $B$ are homotopy equivalent as C*-algebras, then

$$
grs(A) = grs(B), \text{ and } csr(A) = csr(B)$$

This makes it both interesting, and hard to compute, because it is not well-behaved with respect to various natural constructions (ideals, quotients, extensions, etc.).

**Example**

$$C(D^n) \twoheadrightarrow C(S^{n-1})$$

but $grs(C(D^n)) = 1$ and $grs(C(S^{n-1})) \neq 1$ for $n > 5$. 
Computing $gsr(C(\mathbb{S}^n))$

Let $A = C(\mathbb{S}^n)$, $P$ a projective module over $A$ such that

$$P \oplus A \cong A^m$$

- $P$ corresponds to a vector bundle over $\mathbb{S}^n$ of rank $(m-1)$.
- This vector bundle corresponds (via the clutching construction) to a map $f : \mathbb{S}^{n-1} \to GL_{m-1}(\mathbb{C})$. In fact,

$$\text{Vect}_{m-1}(\mathbb{S}^n) \leftrightarrow [\mathbb{S}^{n-1}, GL_{m-1}(\mathbb{C})]$$

$$= \pi_{n-1}(GL_{m-1}(\mathbb{C}))$$

- Since $P \oplus A \cong A^m$, $\theta(f)$ is null-homotopic, where

$$\theta(f)(x) := \begin{pmatrix} f(x) & 0 \\ 0 & 1 \end{pmatrix} \in GL_m(\mathbb{C})$$
Hence, $\text{gsr}(C(S^n))$ is the least $k \geq 1$ such that, for all $m \geq k$, the map

$$\theta : \pi_{n-1}(GL_{m-1}(\mathbb{C})) \to \pi_{n-1}(GL_m(\mathbb{C}))$$

is injective.

**Theorem (Nica, 2011)**

$$\text{gsr}(C(S^n)) = \begin{cases} 
1 & : n \leq 4 \\
\lceil \frac{n}{2} \rceil & : n > 4, n \in 4\mathbb{Z} \\
\lceil \frac{n}{2} \rceil + 1 & : n > 4, n \notin 4\mathbb{Z}
\end{cases}$$

The proof uses existing knowledge about the groups $\pi_\ell(GL_k(\mathbb{C}))$. 
Generalized Clutching Construction

Extending this clutching construction, we have

**Theorem**

If \( A = C(\Sigma X) \otimes B \) as above. Every projective module over \( A \) can be described using two input data:

1. A projective module \( V \) over \( B \)
2. A continuous function \( u : X \rightarrow \text{Aut}_B(V) \)

In fact, there is an explicit formula

\[
M(V, u) := \{ \varphi : [0, 1] \times X \rightarrow \text{Aut}_B(V) : \varphi(0, x) = \varphi(s, x_0) \text{ and } \varphi(1, x) = u(x)\varphi(0, x) \}
\]
Define $\text{inj}_X(B)$ to be the least $n \geq 1$ such that

$$\theta : [X, GL_{m-1}(B)] \rightarrow [X, GL_m(B)]$$

is injective for all $m \geq n$.

**Theorem**

$$\text{gsr}(C(\Sigma X) \otimes B) = \max\{\text{gsr}(B), \text{inj}_X(B)\}$$
Given two \(\ast\)-homomorphisms \(\delta : B \to D\), \(\gamma : C \to D\), we define the pullback

\[
A = \{(b, c) \in B \oplus D : \delta(b) = \gamma(c)\}
\]

This is described by a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow{\beta} & & \downarrow{\delta} \\
C & \xrightarrow{\gamma} & D
\end{array}
\]

where \(\alpha(b, c) = b\), and \(\beta(b, c) = c\).
Theorem

Given a pullback

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \gamma & \rightarrow D
\end{array}
\]

where either \(\gamma\) or \(\delta\) is surjective, then

\[
gsr(A) \leq \max\{csr(B), csr(C), gsr(C(\mathbb{T}, D))\}
\]

\[
csr(A) \leq \max\{csr(B), csr(C), csr(C(\mathbb{T}, D))\}
\]

If \(K_1(D) = 0\), then the first inequality can be improved to

\[
gsr(A) \leq \max\{gsr(B), gsr(C), gsr(C(\mathbb{T}, D))\}
\]
Examples
If $X = X_0 \cup_\varphi \mathbb{D}^n$ is a CW-complex where $\varphi : \mathbb{S}^{n-1} \to X_0$ is the attaching map, then $C(X)$ is a pullback

\[
\begin{array}{ccc}
C(X) & \longrightarrow & C(X_0) \\
\downarrow & & \downarrow \\
C(\mathbb{D}^n) & \xrightarrow{\gamma} & C(\mathbb{S}^{n-1})
\end{array}
\]

where $X_0$ is a CW-complex, and $\gamma$ is the restriction map. Thus, $C(X)$ is an \textit{iterated} pullback.
A non-commutative CW-complex is defined inductively by

1. $A_0$ is finite dimensional.
2. Having defined $\{A_0, A_1, \ldots, A_k\}$, we define

$$
\begin{array}{ccc}
A_{k+1} & \longrightarrow & A_k \\
\downarrow & & \downarrow \\
C(\mathbb{D}^n) \otimes F_k & \xrightarrow{\gamma} & C(S^{n-1}) \otimes F_k
\end{array}
$$

where $F_k$ is finite dimensional, and $\gamma$ is the restriction map.

The topological dimension of $A_k$ is defined to be $k$.

If $A_0 = F_k = \mathbb{C}$, this defines a CW-complex.
Theorem
Let $A$ be a NCCW complex of topological dimension $\leq n$, then
\[ csr(A) \leq \left\lceil \frac{n}{2} \right\rceil + 1 \]

Taking inductive limits

Corollary (Nistor, 1987)
If $X$ is a compact Hausdorff space of dimension $\leq n$
\[ csr(C(X)) \leq \left\lceil \frac{n}{2} \right\rceil + 1 \]
$K_1$-bijective $C^*$-algebras

Let $\mathcal{F}$ be the class of $C^*$-algebras $A$ such that

$$\theta_A : GL_{k-1}(A) \to GL_k(A)$$

is a weak homotopy equivalence for all $k \geq 2$. Some examples:

- Non-Commutative Irrational Torus
- Purely infinite, simple $C^*$-algebras
- Unital, $\mathcal{Z}$-stable $C^*$-algebras. ($\mathcal{Z} =$ Jiang Su Algebra)
- $O_n \otimes B$ for any $C^*$-algebra $B$.
- $A \otimes B$ where $A$ is an infinite dimensional, simple $AF$-algebra, and $B$ is any $C^*$-algebra.
Theorem

The class $\mathcal{F}$ is closed under the following operations:

- Extensions
- Tensor product by commutative C*-algebras
- Inductive limits
- Surjective Pullbacks (as above)
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The class $\mathcal{F}$ is closed under the following operations:

- Extensions
- Tensor product by commutative C*-algebras
- Inductive limits
- Surjective Pullbacks (as above)

Theorem

Let $X$ be a compact Hausdorff space and $A \in \mathcal{F}$, then

\[
gsr(C(X) \otimes A) = gsr(A)
\]

\[
csr(C(X) \otimes A) = \begin{cases} 
    csr(A) & \text{if } csr(A) \geq 2 \\
    1 \text{ or } 2 & \text{if } csr(A) = 1
\end{cases}
\]
Let $A$ be a unital C*-algebra. Write

$$\text{Prim}(A) = \{\text{kernel of irreducible representations of } A\}$$

$$\text{Prim}_n(A) = \{\text{kernel of } n\text{-dimensional irr. reps. of } A\}$$

Prim$(A)$ is compact w.r.t. the Jacobson topology.
Let $A$ be a unital C*-algebra. Write
\[
\text{Prim}(A) = \{ \text{kernel of irreducible representations of } A \}
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\]

Prim$(A)$ is compact w.r.t. the Jacobson topology.

For $x \in \text{Prim}(A)$, write $A_x := A/x$. For $a \in A$, write $a(x)$ for its image in $A/x$.

**Definition**

$A$ is called a C*-bundle if, for each $a \in A$, the map
\[
x \mapsto \|a(x)\|
\]
is continuous.
SubHomogeneous C*-algebras

The following theorem was proved by Phillips (~ 2001), but our proof is much easier.

Theorem

Assume that

- $\exists N \in \mathbb{N}$ such that, for each $x \in \text{Prim}(A)$, $\dim(A_x) \leq N$
- $\text{Prim}(A)$ is a manifold and $\dim(\text{Prim}_n(A)) \leq 2n - 1$ for all $n \leq N$

Then $g_{sr}(A) = 1$, and $c_{sr}(C(T, A)) \leq 2$, so that

$$K_1(A) \cong \pi_0(GL_1(A))$$
Bundles of Kirchberg algebras

**Theorem**

Assume that

- For each $x \in \text{Prim}(A)$, $A_x$ is a Kirchberg algebra such that $[A_x]$ has infinite order in $K_0(A_x)$.
- Prim$(A)$ is metrizable, and has finite covering dimension.

Then

\[ gsr(A) = 2, \quad \text{and} \]

\[ csr(C(\mathbb{T}, A)) \leq 2, \quad \text{so that} \]

\[ K_1(A) \cong \pi_0(GL_1(A)) \]
Thank you!