MTH 101 Homework 1

Solutions

1. If \( x \) and \( y \) are arbitrary real numbers with \( x < y \), prove that there is at least one real number \( z \) satisfying \( x < z < y \).

   **Solution:** Let \( z = (x + y)/2 \). We will now show that this \( z \) lies between \( x \) and \( y \). Clearly, \((y - z) = (y - x)/2 > 0\), hence \( z < y \). Similarly, \((z - x) = (y - x)/2\), which implies that \( x < z \).

2. If \( x \) is an arbitrary real number, prove that there exist integers \( m \) and \( n \) such that \( m < x < n \).

   **Solution:** Assume first that there is no integer \( n \) such that \( x < n \). This means that \( n \leq x \) for all integers \( n \), and hence \( x \) is an upper bound for the set of integers. This is a contradiction since we showed in the class that the set of integers is unbounded above. The other case can be worked out in a similar way.

3. If \( x \) is an arbitrary real number, prove that there exists a unique integer \( n \) which satisfies the inequalities \( n \leq x < n + 1 \). (Remark: This unique integer is called the greatest integer in \( x \) and is denoted by \([x]\).)

   **Solution:** Let \( A = \{n \in \mathbb{Z} | n \leq x\} \). By previous problem, \( A \) is non-empty. It is also bounded above, so by completeness axiom, \( s = \sup(A) \) exists. Now, by property of sup, there exists \( n \in A \) such that \( s - 1 < n \leq s \). Clearly, \( n \leq x \). Now, \( s - 1 < n \) implies that \( n + 1 \) does not belong to \( A \). Hence \( x < n + 1 \). This show that there is an \( n \) which satisfies \( n \leq x < n + 1 \). Now, to show uniqueness, assume on the contrary that there exists \( m \neq n \) such that \( m \leq x < m + 1 \). Then \( m \in A \) and we must have \( m \leq n \) (otherwise, if \( m > n \), then \( m \geq n + 1 > x \), so \( m \) cannot satisfy the said inequality). If \( m < n \) then (since both are integers) \((n - m) \geq 1\). Hence, \( m + 1 \leq n \leq x \), which contradicts the assumption that \( m \) satisfies the given inequality.

4. If \( x \) and \( y \) are arbitrary real numbers such that \( x < y \), prove that there exists at least one rational number \( r \) satisfying \( x < r < y \), and hence infinitely many.

   **Solution:** Since \((y - x) > 0\), by the Archimedean property, there exists a positive integer \( n \) such that \( n(y - x) > 1 \). Now, let \( m = [nx] \) (the greatest integer \( \leq nx \)). Then \( m \leq nx < m + 1 \). Also, we must have \( m + 1 < ny \). To see this, assume that \( m + 1 \geq ny \). Then, \( ny - 1 \leq m \leq nx \), which is a contradiction since \( n(y - x) > 1 \). Thus, we have \( m \leq nx < m + 1 < ny \). Dividing by \( n \), we get \((m/n) \leq x < (m + 1)/n < y \). We have thus proved that the rational number \( r = (m + 1)/n \) lies between \( x \) and \( y \). We can now apply this result to \( r \) and \( y \).

5. Is the sum or product of two irrational numbers always irrational?

   **Solution:** No. The sum or product of two irrational numbers may be rational or irrational. For example, \((\sqrt{2})(\sqrt{2}) = 2\), which is rational, whereas \(\sqrt{2}\sqrt{3} = \sqrt{6}\), which is irrational. Likewise, \(\sqrt{2} + \sqrt{2} \) is irrational, whereas \((\sqrt{2} - \sqrt{2}) = 0\) is rational.

6. If \( x \) and \( y \) are arbitrary real numbers such that \( x < y \), prove that there exists at least one irrational number \( z \) satisfying \( x < z < y \), and hence infinitely many.

   **Solution:** We know by Problem 4 that there exists a rational number between two real numbers. Hence, there exists a rational number \( q \) such that \( \frac{x}{\sqrt{2}} < q < \frac{y}{\sqrt{2}} \) and \( z = q\sqrt{2} \) is an irrational number between \( x \) and \( y \) (here we have used the irrationality of \( \sqrt{2} \)).

7. An integer \( n \) is called even if \( n = 2m \) for some integer \( m \), and odd if \( n + 1 \) is even. Prove the following statements:

   (a) An integer cannot be both even and odd.
(b) Every integer is either even or odd.

(c) The sum or product of even integers is an even integer. What can you say about the sum or product of odd integers?

(d) If $n^2$ is even, so is $n$. If $a^2 = 2b^2$, where $a$ and $b$ are integers, then both $a$ and $b$ are even.

(e) Every rational number can be expressed in the form $a/b$ where $a$ and $b$ are integers, at least one of which is odd.

Solution:

(a) If an integer is both even and odd, there exist integers $m$ and $n$ such that $2m = 2n + 1$. This implies that $1 = 2(m−n)$, which means that 1 is even but this is a contradiction since 1(= 0+1) is odd. Hence every integer is either even or odd.

(b) Suppose on the contrary that the set 
$$S = \{k \in \mathbb{Z} | k \text{ is neither even nor odd}\}$$

is non-empty. Now, we know that $T = S^c$ (complement of $S$ in $\mathbb{Z}$) is non-empty. Choose $n \in S$ such that $t := n−1 \in T$. Such an $n$ exists because $\mathbb{Z}$ is an inductive set. If $t$ is even, this means that $n$ is odd; if $t$ is odd then $n$ is even. In either case, we get a contradiction since we assumed that $n \in S$.

(c) We have $(2n)(2m) = 4nm$ and $2n + 2m = 2(n + m)$, so product and sum of even integers is even. Also, $(2m+1)(2n+1) = 2(2mn+m+n)+1$, so the product of odd integers is odd. Finally, $(2n+1)+(2m+1) = 2(n + m + 1)$, so the sum of odd integers is even.

(d) This follows from part (c) above.

(e) Assume that $r = p/q$ has been expressed in the reduced form. If $p$ and $q$ both are even, then there exist integers $n$ and $m$ such that $p = 2n$ and $q = 2m$. But this means that $p$ and $q$ have a common factor. This contradicts the assumption that $p/q$ is in reduced form. Hence, at least one of $p$ or $q$ must be odd.

8. Prove that there is no rational number whose square is 2.

Solution: This is a consequence of Problem 7(d).