# Electronic properties of solids - Drude Model

*Lecture 11*

# CHM 637

# Chemistry & Physics of Materials

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# Lecture Plan

- Basic Assumptions of Drude model
- DC conductivity

# Basic Assumptions of Drude Model

1. Conduction in metals is due to freely moving valence electrons.
2. The ions in the metal are much heavier and hence immobile.
3. The free electrons can be treated through kinetic theory of gases by assuming they are ideal gas particles.
   (Despite much larger densities of ~ $10^{22}$ per c.c. !!!)
4. The electrons undergo collisions (with the lattice ions) which randomises their velocities and also is responsible for achieving thermal equilibrium with surroundings.

Here, "randomises" implies that the velocity of an electron after a collision has absolutely no correlation with its velocity before the collision either in direction or magnitude.

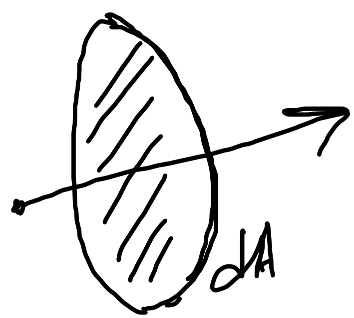
# Basic Assumptions of Drude Model

5. Between collisions the interaction of the electron with other electrons and ions is neglected => in the absence of external fields, between collisions each electron moves in a straight line.
6. Collisions are instantaneous events that abruptly change the velocity of an electron.
7. An electron experiences a collision in an interval $t$ to $t+dt$ with a probability $dt/\tau$ where $\tau$ is defined as the "relaxation time" or the mean time between successive collisions. i.e. $\langle t \rangle = \tau$

# Drude Model for conduction in Metals

$\vec{j}(\vec{r}) \longrightarrow$ current density = current per unit area

current = charge flowing in a given direction per unit time



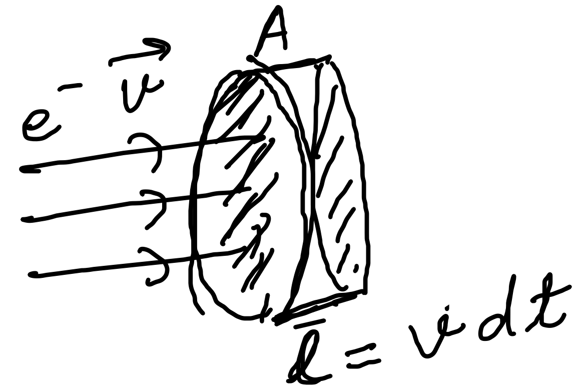

$$\vec{j}(\vec{r}) = \sigma \vec{E}(\vec{r})$$

Local form of the Ohm's Law

$\sigma = 1/\rho \rightarrow$ conductivity ($S\,m^{-1}$)

$S \rightarrow \Omega^{-1}$.

Current = charge / time




no. density of conduction e's

$$= \frac{n \times (A v\,dt)(-e)}{dt}$$

$$= -nevA$$

Current density $= j = \dfrac{\text{current}}{\text{Area}} = -nev$

average

$$\boxed{\vec{J}(\vec{r}) = -ne\langle \vec{v}(\vec{r}) \rangle}$$

$$\boxed{\vec{j} = -ne\,\vec{v}_{avg.}}$$

Let's assume that an election undergoes a series of collisions labeled by $j$

Let the velocity of the election right after the $j^{th}$ collision be $\vec{v}_j$

Remember that $\langle \vec{v}_j \rangle = 0$

$\therefore$ in the absence of an external electric field the average electronic velocity between 2 successive collision is $\vec{v}_{avg} = \langle \vec{v}_j \rangle = 0$.

$$\Rightarrow \vec{j} = 0 \quad (\text{w/o external } \vec{E})$$

In the presence of an ext. electric field $\vec{E}$, we have for the velocity between collisions $j$ & $j+1$ :

$$t_j < t < t_{j+1}$$

$$\vec{V}(t) = \vec{V}_j - \underbrace{\left(\frac{e\vec{E}}{m}\right)}(t - t_j)$$

acceleration felt by the $e^-$ due to ext. field

$$\vec{v} = \vec{u} + \vec{a}t$$

$$\vec{V}_{avg} = \langle \vec{v}(t) \rangle = \underbrace{\langle \vec{v}_j \rangle}_{=0} - \frac{e\vec{E}}{m} \underbrace{\langle (t - t_j) \rangle}_{\tau}$$

$\tau$ = average time between 2 succ. collisions

$$= -\frac{e\vec{E}}{m}\tau$$

$$\Rightarrow \boxed{\vec{j} = \underbrace{\left(\frac{ne^2\tau}{m}\right)}_{\sigma = \frac{ne^2\tau}{m}} \vec{E}}$$

$$\mu = \frac{|\langle \vec{v} \rangle|}{|\vec{E}|} \rightarrow \text{mobility of the electron}$$

$$= \frac{e\tau}{m}$$

$$\vec{j} = ne\mu\vec{E}$$

Drude conductivity $\boxed{\sigma = \frac{ne^2\tau}{m}}$

$$\tau = \frac{m}{ne^2\rho}$$ } exptly. measurable parameters

theoretical parameter

$$n = \frac{Z \times \rho_m \times 6.022 \times 10^{23}}{A}$$

# Basic Assumptions of Drude Model

Typical relaxation times computed from resistivity data

DRUDE RELAXATION TIMES IN UNITS OF $10^{-14}$ SECOND[a]

| ELEMENT | 77 K | 273 K | 373 K |
|---|---|---|---|
| Li | 7.3 | 0.88 | 0.61 |
| Na | 17 | 3.2 | |
| K | 18 | 4.1 | |
| Rb | 14 | 2.8 | |
| Cs | 8.6 | 2.1 | |
| Cu | 21 | 2.7 | 1.9 |
| Ag | 20 | 4.0 | 2.8 |
| Au | 12 | 3.0 | 2.1 |
| Be | | 0.51 | 0.27 |
| Mg | 6.7 | 1.1 | 0.74 |
| Ca | | 2.2 | 1.5 |
| Sr | 1.4 | 0.44 | |
| Ba | 0.66 | 0.19 | |
| Nb | 2.1 | 0.42 | 0.33 |
| Fe | 3.2 | 0.24 | 0.14 |
| Zn | 2.4 | 0.49 | 0.34 |
| Cd | 2.4 | 0.56 | |
| Hg | 0.71 | | |
| Al | 6.5 | 0.80 | 0.55 |
| Ga | 0.84 | 0.17 | |
| In | 1.7 | 0.38 | 0.25 |
| Tl | 0.91 | 0.22 | 0.15 |
| Sn | 1.1 | 0.23 | 0.15 |
| Pb | 0.57 | 0.14 | 0.099 |
| Bi | 0.072 | 0.023 | 0.016 |
| Sb | 0.27 | 0.055 | 0.036 |

Source: Ashcroft and Mermin

# Drude model for conduction in metals

$$\vec{j} = \sigma \vec{E} \quad \text{(Ohm's Law)}$$

$$\sigma_0 = \frac{ne^2 \tau}{m} \quad \text{(DC conductivity)}$$

$$\mu = \frac{\langle v \rangle}{E} = \frac{e\tau}{m}$$

$v_0 \sim 10^7$ cm/s

$l \sim 1\text{-}10\,\mathring{A}$

$$\tau = \frac{m}{e^2 n}\left(\frac{1}{\rho}\right) \sim 10^{-14} \text{ to } 10^{-15}\,s$$

$v_0 \tau \simeq l \rightarrow$ mean free path

$$\vec{v}(t) = \vec{p}(t)/m$$

total momentum per electron

$$\vec{j} = -\frac{ne\,\vec{p}(t)}{m}$$

Let total momentum per electron at $t+dt$

$$= \vec{p}(t+dt)\ \cancel{s}.$$

Consider an election at random, it will have a collision between $t$ and $t+dt$ w/ the probability $= dt/\tau$.

$\therefore$ The probability that in the same time interval it will not have a collision $= 1 - dt/\tau$

Prob. of no collision $= \dfrac{\text{No. of elections undergoing no collission}}{N}$

Momentum per electron

$$\vec{p}(t+dt) = \vec{p}(t) + \vec{f}(t)\,dt + O(dt)^2$$

force due to external fields applied

$$\vec{p}^{\,\text{no-coll}}(t+dt) = \left(1-\frac{dt}{\tau}\right)\left[\vec{p}(t) + \vec{f}(t)\,dt + O(dt)^2\right]$$

$$\vec{p}^{\,\text{coll}}(t+dt) = \left(\frac{dt}{\tau}\right)\left[\vec{f}(t)\,dt + O(dt)^2\right]$$

$\langle p(t)\rangle \to 0.$

$$= O(dt)^2.$$

$$\therefore \vec{p}(t+dt) = \vec{p}^{\,no\text{-}coll}(t+dt) + \vec{p}^{\,coll}(t+dt)$$

$$= \left(1 - \frac{dt}{\tau}\right)\left[\vec{p}(t) + f(t)\,dt\right] + O(dt)^2$$

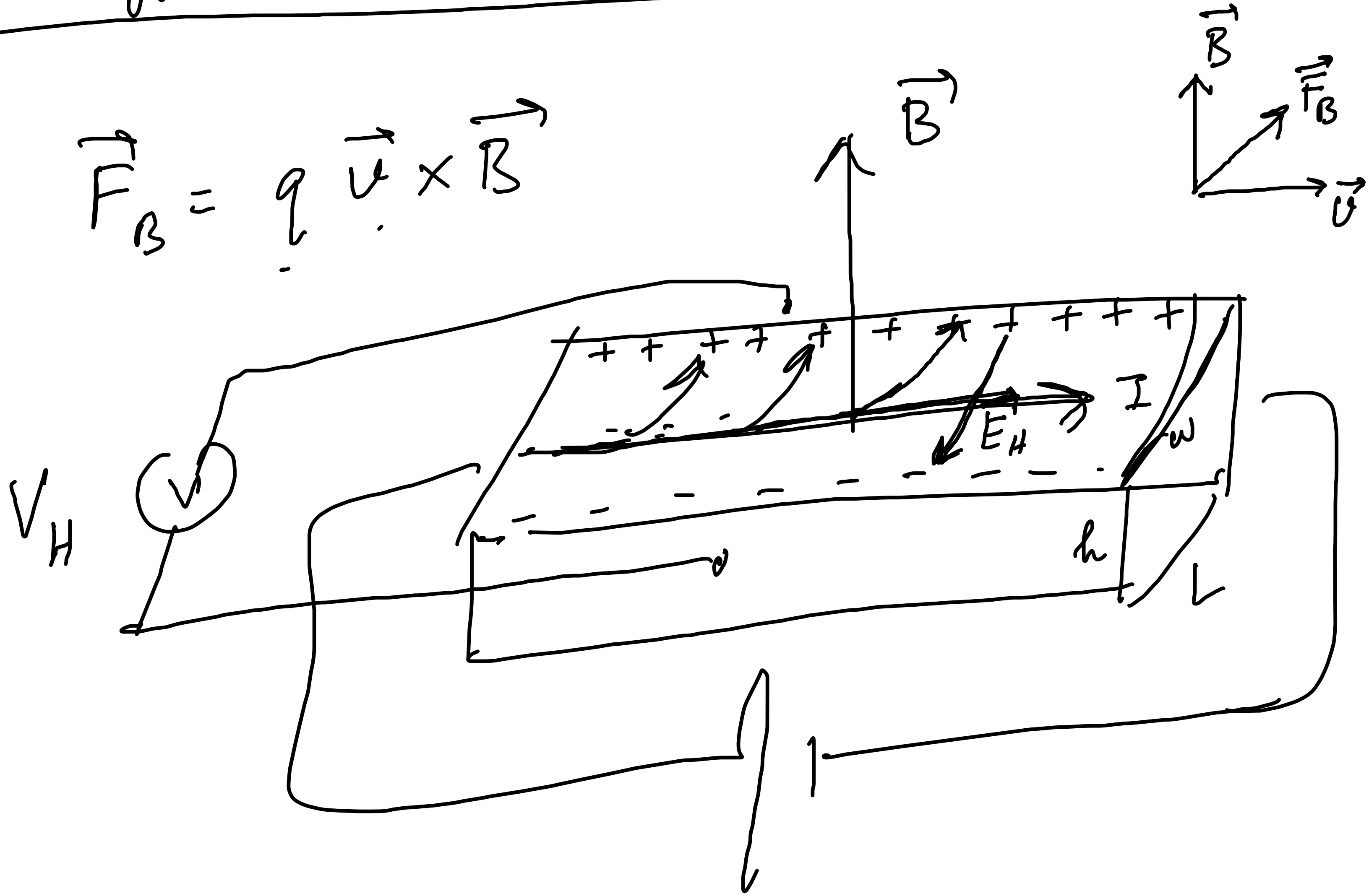
$$\simeq \vec{p}(t) + \vec{f}(t)\,dt - \frac{dt}{\tau}\vec{p}(t)$$

$$\lim_{dt\to 0}\left(\frac{\vec{p}(t+dt) - \vec{p}(t)}{dt}\right) = \frac{d\vec{p}(t)}{dt} = -\frac{1}{\tau}\vec{p}(t) + f(t)$$

$$\therefore \boxed{\frac{d\vec{p}}{dt} = \underbrace{-\frac{\vec{p}(t)}{\tau}}_{\text{damping}} + \underbrace{\vec{f}(t)}_{\text{driving force}}}$$

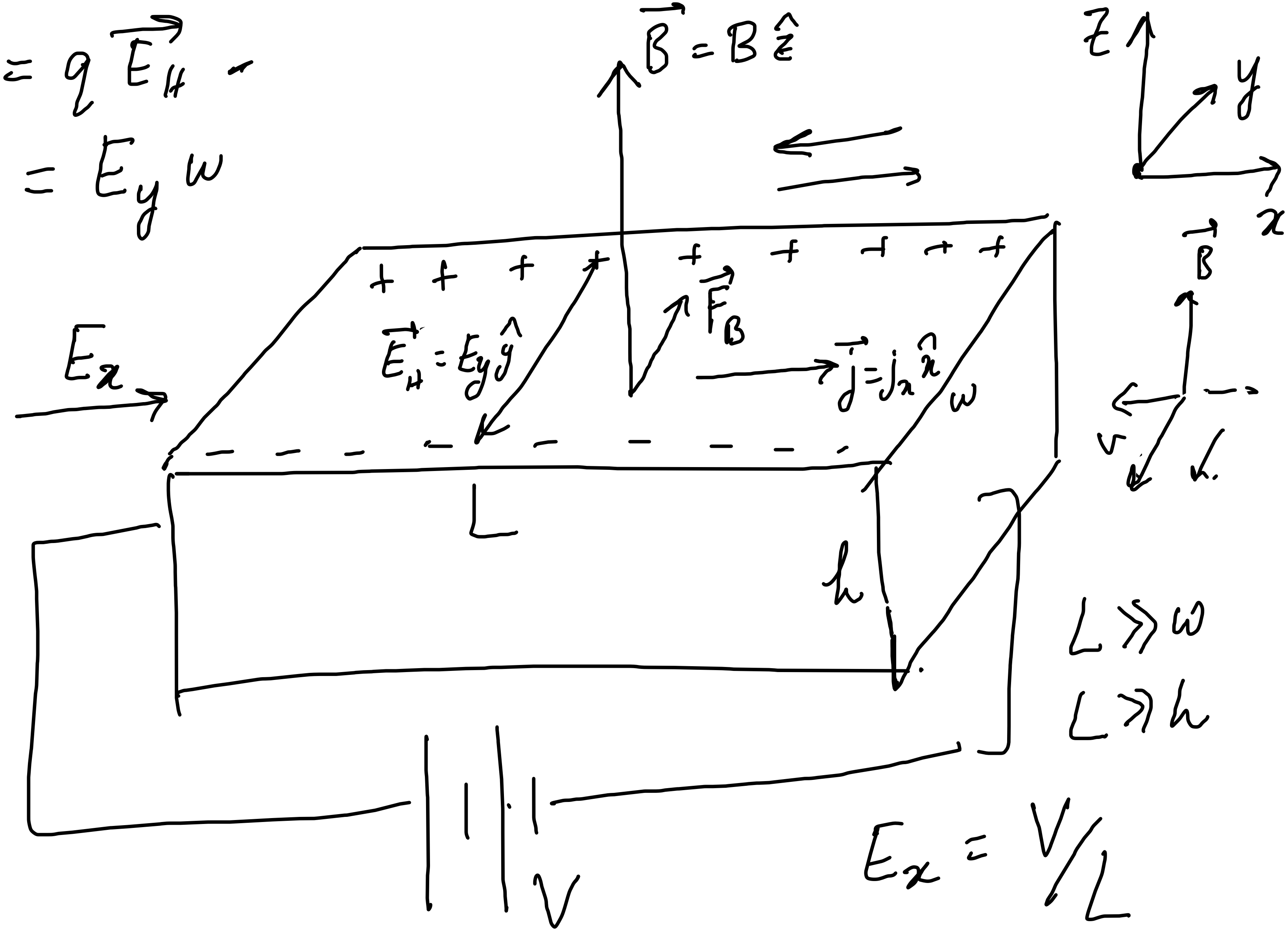
# Hall effect & Magnetoresistance : (Hall 1879)

$$\vec{F}_B = q\,\vec{v} \times \vec{B}$$



$$\vec{F}_H = q\,\vec{E}_H$$

$$V_H = E_y\, w$$




$L \gg w$

$L \gg h$

$$E_x = V/L$$

Magnetoresistivity : $\rho(B) = \dfrac{E_x}{j_x}$

Hall coefficient : $R_H = \dfrac{E_y}{j_x B}$

Note that $E_y$ & $j_x$ have opposite signs for electrons and same sign for "holes" or positive charges flowing along $\hat{x}$.

$$R_H \begin{cases} < 0 & \text{for electrons} \\ > 0 & \text{for holes.} \end{cases}$$

Gives the sign of the charge

$\vec{j}, \vec{E} = ?$ in presence of $\vec{B}$

$$\vec{f} = -e\left(\vec{E} + \vec{v} \times \vec{B}\right) \rightarrow \text{Lorentz force}$$

$$\frac{d\vec{p}}{dt} = -\frac{\vec{p}}{\tau} - e\left(\vec{E} + \frac{\vec{p}}{m} \times \vec{B}\right).$$

At steady state, $\frac{d\vec{p}}{dt} = 0$

$$0 = -\frac{p_x}{\tau} - e\left(E_x + \frac{p_y B}{m}\right)$$

$$= -p_x/\tau - eE_x - \omega_c p_y$$

$$0 = -p_y/\tau - eE_y - \omega_c p_x$$

multiply by $-\frac{ne\tau}{m}$

$$\omega_c = \frac{eB}{m}$$

Cyclotron frequency

$$\sigma_0 E_x = \omega_c \tau j_y + j_x ,$$

$$\sigma_0 E_y = -\omega_c \tau j_x + j_y ,$$

$\sigma_0 = ne^2\tau/m$ Drude conductivity

Hall field is determined by claiming balance between $\vec{F}_B$ & $\vec{F}_H$. $\Rightarrow j_y = 0$

$$E_y = \left(-\frac{\omega_c \tau}{\sigma_0}\right) j_x = -\left(\frac{B}{ne}\right) j_x$$

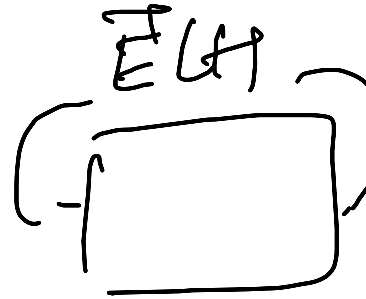
$$E_y = -\left(\frac{B}{ne}\right) j_x \Rightarrow R_H = \frac{E_y}{j_x B} = -\frac{1}{ne}$$

& $$\sigma_0 E_x = j_x \Longrightarrow \rho(B) = \frac{E_x}{j_x} = \frac{1}{\sigma_0}$$

$$= \rho_0$$

Drude model indicates no magnetic field dependence of resistivity & hence trivial result

# AC conductivity in metals




Let $\vec{E}(t) = \mathrm{Re}\left\{E_0(\omega)\, e^{-i\omega t}\right\}$

(spatially uniform)

Let's look for solutions for $\vec{p}(t)$ of the form $\vec{p}(t) = \mathrm{Re}\left\{\vec{p}(\omega)\, e^{-i\omega t}\right\}$

$$\frac{d\vec{p}(t)}{dt} = -\frac{\vec{p}(t)}{\tau} - e\vec{E}(t)$$

$$\Rightarrow \; -i\omega \vec{p}(\omega) = \frac{-\vec{p}(\omega)}{\tau} - e\vec{E}(\omega)$$

since $\vec{j} = -ne\vec{p}/m$, $\vec{j}(t) = Re\left[\vec{j}(\omega)e^{-i\omega t}\right]$

$$\therefore \; \vec{j}(\omega) = \frac{-ne\vec{p}(\omega)}{m}$$

$$\vec{p}(\omega) = \frac{-e\vec{E}(\omega)}{\left(1/\tau - i\omega\right)}$$

$$\Rightarrow \; \vec{j}(\omega) = \left(ne^2/m\right)\vec{E}(\omega) \Big/ \left(\frac{1}{\tau} - i\omega\right)$$

$$\vec{j}(\omega) = \underbrace{\left(\frac{\sigma_0}{1 - i\omega\tau}\right)}\vec{E}(\omega)$$

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}}$$

$\sigma_0 \rightarrow$ Drude conductivity

Most important application of this formula is in the propagation of e.m. waves in metals.

Maxwell's eqns: (in charge-free medium)

$\rho = 0 \rightarrow$ charge density

$$\vec{\nabla} \cdot \vec{E} = 0$$ (Gauss law) SI units

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

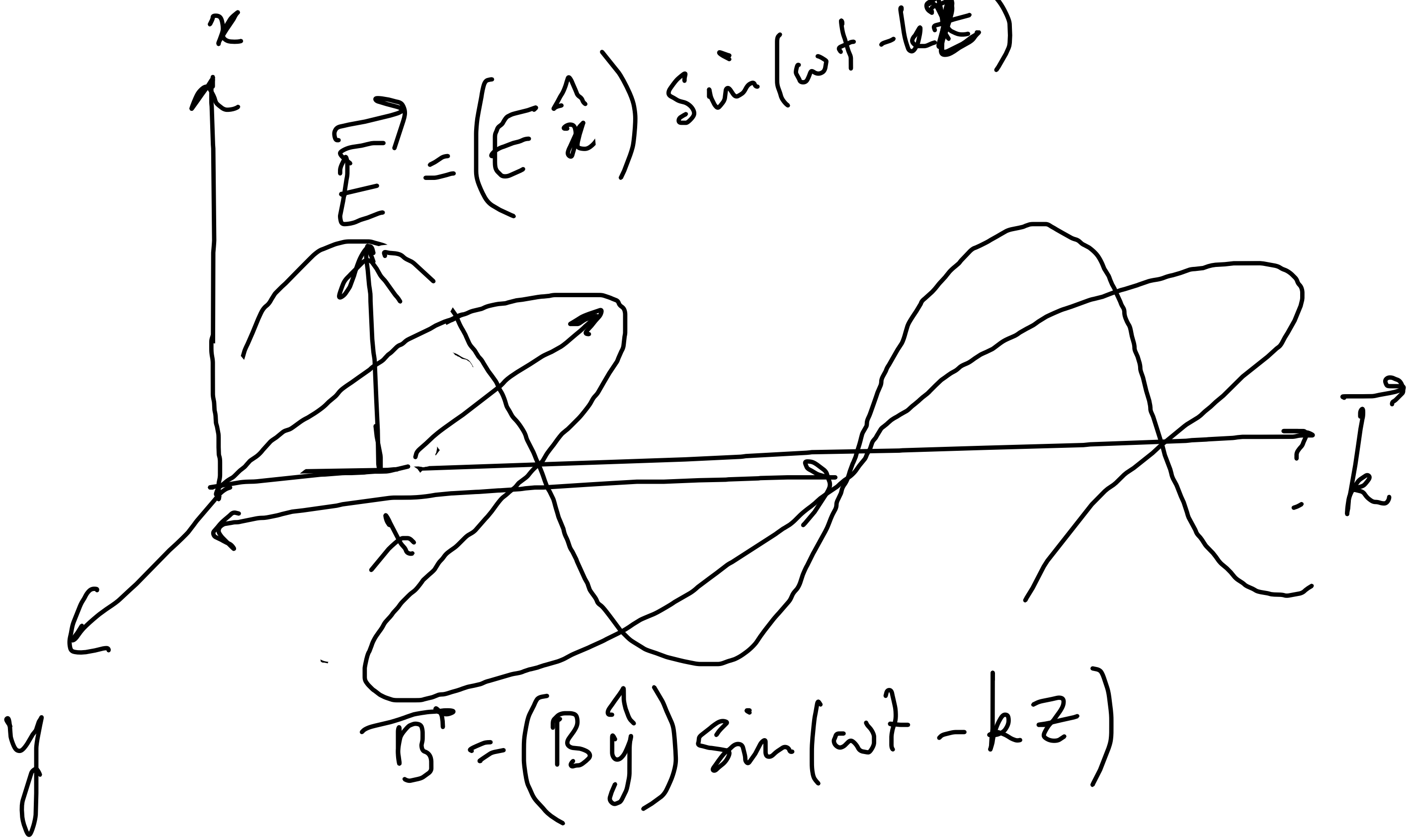
$3 \times 10^8$ m/s

$$c^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$ (no magnetic monopole)

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$ (Ampere's Circuital Law)

$$\vec{E} = (E\hat{x})\sin(\omega t - kz)$$




$$\vec{B} = (B\hat{y})\sin(\omega t - kz)$$

2 complications:

1) We ignored the magnetic field effects. This is justified since the relevant M.F. terms are suppressed by a factor $\frac{v}{c}$ which is usually very small.

2) We assumed spatially uniform fields. $|\vec{k}| = 2\pi/\lambda$. If wavelength of light used is $\gtrsim$ mean-free-path of collisions then spatial variation can be ignored.

To account for spatial variation of field we can write :

$$\vec{J}(\vec{r},\omega) = \sigma(\omega)\,\vec{E}(\vec{r},\omega) \; .$$

$$(\lambda \gg l)$$

$$\lambda \sim 10^{3} - 10^{4}\,\mathring{A}$$

Let's first look at the situation when

$\rho = 0$

$$\vec{\nabla} \times \overbrace{\vec{\nabla} \times \vec{E}} = -\frac{\partial\, \vec{\nabla} \times \vec{B}}{\partial t}$$

$$= -\mu_0 \frac{\partial \vec{j}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t} = -\nabla^2 \vec{E}$$

$$-\nabla^2 \vec{E}(\omega) = \mu_0 i \omega \vec{j}(\omega) + \mu_0 \epsilon_0 \omega^2 \vec{E}(\omega)$$

$$= \boxed{\frac{\omega^2}{c^2} \underline{\underline{\epsilon(\omega)}}} \vec{E}(\omega)$$

where $\epsilon(\omega) = \left\{ 1 + i \frac{\sigma(\omega)}{\epsilon_0 \omega} \right\}$

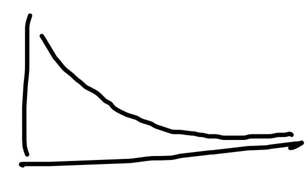
$$\nabla^2 \vec{E} = -\overbrace{\left(\frac{\omega^2}{c^2}\right)}^{k^2}\vec{E}(\omega)$$

$$-k^2\vec{E}(\vec{k},\omega) = -\frac{\omega^2}{c^2}\vec{E}(\vec{k},\omega)$$

$$k = \omega/c \iff c = \nu\lambda$$

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\vec{k} = i\vec{K} \longrightarrow e^{-\vec{K}\cdot\vec{r}}$$



In our case, $\boxed{k^2 = \frac{\omega^2}{c^2}\epsilon(\omega)}$

or $\quad k = \omega/c \; \epsilon(\omega)^{1/2}$

$$\epsilon(\omega) = 1 + i\frac{\sigma(\omega)}{\epsilon_0 \omega}$$

$\omega_p \to$ Plasma frequency

Assume that $\omega\tau \gg 1$,

where $\omega_p^2 = \frac{ne^2}{m\epsilon_0}$.

$$\boxed{\epsilon(\omega) \simeq 1 - \omega_p^2/\omega^2}$$

$$k^2 = \frac{\omega^2}{c^2}\,\epsilon(\omega)$$

$$\text{if} \quad \epsilon(\omega) \begin{cases} \text{real \& positive} \Rightarrow k \text{ is real} \\ \text{real \& negative} \Rightarrow k \text{ is imaginary} = i\underline{K} \end{cases}$$

$$\vec{k} = \underline{k}\,\hat{z}$$

$$\rightarrow e^{i\vec{k}\cdot\vec{r}} \rightarrow e^{-Kz}$$

$$\epsilon(\omega) \simeq 1 - \frac{\omega_p^2}{\omega^2} \qquad (\omega\tau \gg 1)$$

$$\Rightarrow \quad \epsilon(\omega) = \begin{cases} < 0 & \text{if } \omega < \omega_p \\ > 0 & \text{if } \omega > \omega_p \end{cases}$$

$\omega_p$ decides whether or not e.m. wave propagates through a metal