CHM 428/638 Assignment 1

January 28, 2024

Due on 6^{th} Feb, 2024.

1. In the following expressions where $\hat{A}$ is an operator, specify whether each expression is a ket, a bra, an operator or a number.
   (a) $\langle\phi|\hat{A}|\psi\rangle\langle\psi|$
   (b) $\hat{A}|\psi\rangle\langle\psi|$
   (c) $\langle\phi|\hat{A}|\psi\rangle\langle\psi|\hat{A}$
   (d) $|\psi\rangle\langle\phi|$
   (e) $\langle\psi|\hat{A}$
   (f) $\hat{A}|\phi\rangle$

2. For a linear operator $\hat{A}$ show that
   (a) For any two states $|\phi\rangle$ and $|\psi\rangle$, $\langle\psi|\hat{A}^\dagger|\phi\rangle = \overline{\langle\phi|\hat{A}|\psi\rangle}$.
   (b) $\left(\hat{A}^\dagger\right)^\dagger = \hat{A}$.
   (c) $\left(\hat{A}_1\hat{A}_2\ldots\hat{A}_n\right)^\dagger = \hat{A}_n^\dagger\hat{A}_{n-1}^\dagger\ldots\hat{A}_1^\dagger$.

3. Show that any linear operator can be written as a sum of Hermitian and a skew-Hermitian operator. If an operator is assumed to be associated with a complex dynamical variable and its adjoint with the complex conjugate of the variable then also show that operators corresponding to real dynamical variables are always Hermitian.

4. The commutator of any two operators $\hat{A}$ and $\hat{B}$ is defined as $\hat{C}(A,B) = \left[\hat{A},\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Show that

   (a) $\hat{C}^\dagger(A,B) = -\hat{C}^\dagger(A^\dagger, B^\dagger)$

   (b) $\hat{C}(A,BD) = \hat{C}(A,B)\hat{D} + \hat{B}\hat{C}(A,D)$

   (c) $\left[\hat{A},\hat{C}(B,D)\right] + \left[\hat{B},\hat{C}(D,A)\right] + \left[\hat{D},\hat{C}(A,B)\right] = 0$ (Jacobi identity).

5. A Hermitian operator $\hat{A}$ has the eigenvalue equation $\hat{A}|\phi_n\rangle = \lambda_n|\phi_n\rangle$. Show that

   (a) All eigenvalues are real and the eigenkets corresponding to two different eigenvalues are mutually orthogonal.

   (b) The eigenkets belonging to the same (degenerate) eigenvalue form a linear vector space of dimension equal to the degeneracy.

   (c) Show that $sin(k\hat{A})|\phi_n\rangle = sin(k\lambda_n)|\phi_n\rangle$.

   (d) If $\hat{A}^m|\psi\rangle = 0$ for some ket $|\psi\rangle$ and some integer $m > 0$ then $\hat{A}|\psi\rangle = 0$.

6. Show that the eigenvalues of a skew-Hermitian operator are either purely imaginary or zero.

7. Show that if two linear operators $\hat{A}$ and $\hat{B}$ commute then $\hat{A}$ and $f(\hat{B})$ also commute where f is a function of the operator $\hat{B}$.

8. If $\hat{A}$ and $\hat{B}$ are two linear Hermitian operators then show that $\hat{A}\hat{B}$ is Hermitian only if $\hat{A}$ and $\hat{B}$ commute.

9. A system was prepared in a quantum state $|\psi\rangle$ on which a measurement of the observable $\hat{P}$ yields the values p_1 with 40% probability and p_2 with 60% probability. Also a subsequent measurement of the observable $\hat{Q}$ results in the values q_1 and q_2 with the probability ratio 2:3 if the result of the first measurement was p_1 and in the ratio 1:4 if the result of the first measurement was p_2.

   (a) Construct (at least) one guess for $|\psi\rangle$ in terms of the eigenstates of $\hat{P}$.

(b) Using the above guess write down a relation between the eigenstates of $\hat{P}$ and $\hat{Q}$. Remember that both operators are Hermitian.

(c) Write down $|\psi\rangle$ in terms of the eigenstates of $\hat{Q}$.

(d) Do the two observables commute?

(e) Construct the matrix $U_{ij} = \langle p_i | q_j \rangle$ and show that it is unitary.