Solutions to Assignment 1

1. (a) $y = \sqrt{16 - x^2}$

RHS must be real for y to be real.

$\Rightarrow \; 16 - x^2 \geq 0 \;\Rightarrow\; x^2 \leq 16$

$\Rightarrow \; -4 \leq x \leq 4$ maximum domain

(b) $y = 1/x^2+8$

All real values of x are allowed here. $\therefore \; x \in \mathbb{R}$. is the maximum domain
or $x \in (-\infty, \infty)$

(c) $y = \ln(x) \;\Rightarrow\; x = e^y > 0$

$\therefore \; x \in (0, \infty)$ maximum domain.

(d) $y = \dfrac{1}{x-1}$ All real values of x are allowed.
Note that at $x=1$ the function is undefined (but tends to a real number)
So this pt. can be excluded. $x \in (-\infty, 1) \cup (1, \infty)$

2. $\tan(y)$ is periodic with period π.

(a) $\Rightarrow \tan(y + m\pi) = \tan(y)$ — ① $\quad m = 0, \pm 1, \pm 2, \dots$

Let $y = 2x$, then ① implies

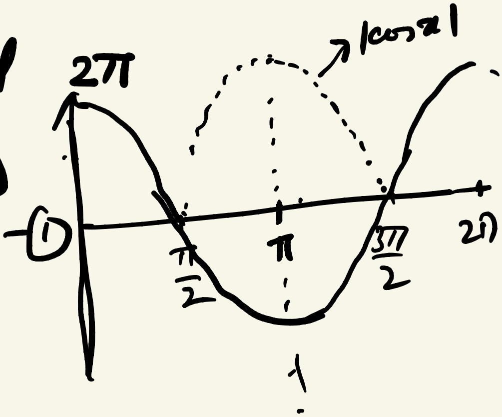
$$\tan(2x + m\pi) = \tan(2x)$$

or $\tan\left(2\left[x + \frac{m\pi}{2}\right]\right) = \tan(2x)$ — ② $\quad m = 0, \pm 1, \pm 2, \dots$

$\Rightarrow$ as a function of x, $\tan(2x)$ is periodic with a period of $\pi/2$

(b) $\cos(x)$ has a period 2π

i.e. $\cos(x + m2\pi) = \cos(x)$ — ① $\quad m = 0, \pm 1, \dots$



Now,

$$\cos(x + \pi) = -\cos x$$

$\therefore \left|\cos(x + \pi)\right| = \left|\cos x\right|$

i.e. $|\cos x|$ has a period π.

(c) $\frac{\sin(x)}{x}$, is not periodic.

3. (a) $\lim_{x\to 0} \frac{\sin(3x)}{x} = 3 \lim_{x\to 0} \frac{\sin(3x)}{(3x)}$

$= 3 \lim_{y\to 0} \frac{\sin(y)}{y} \quad (y = 3x)$

$= \underline{\underline{3}}$ $\quad \because \lim_{y\to 0} \frac{\sin(y)}{y} = 1$ is known.

(b) $\lim_{x\to 0} \frac{\sin(2x)}{\sin(x)} = 2 \lim_{x\to 0} \frac{\left(\frac{\sin(2x)}{2x}\right)}{\left(\frac{\sin x}{x}\right)}$

$= 2 \frac{\lim_{x\to 0} \frac{\sin(2x)}{2x}}{\lim_{x\to 0} \frac{\sin x}{x}} = \underline{\underline{2}}.$

Note that the original ratio has a 0/0 form at $x\to 0$. However, once we divided numerator & denominator by x, we have that the denominator does not go to zero as $x\to 0$

$\therefore \lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)}$ can be used.

(c) $$\underset{x\to 1}{Lt} \frac{1+\cos(\pi x)}{\tan^2(\pi x)} = \underset{x\to 1}{Lt} \frac{(1+\cos(\pi x))}{(1-\sec^2(\pi x))}$$

$$= \underset{x\to 1}{Lt} \frac{(1+\cos(\pi x))}{(\cos^2(\pi x)-1)}$$

$$= \underset{x\to 1}{Lt} \frac{(1+\cos(\pi x))}{(\cos(\pi x)+1)(\cos(\pi x)-1)}$$

$$= \underset{x\to 1}{Lt} \frac{1}{\cos(\pi x)-1}$$

$$= \frac{\underset{x\to 1}{Lt}\ 1}{\left[\underset{x\to 1}{Lt}\cos(\pi x)\right]-1} = -\frac{1}{2}$$ //

Once again, the original ratio was of 0/0 form. So the last step was not possible directly.

(d) $\underset{x \to 0}{Lt} \dfrac{\sqrt{x+2} - \sqrt{2}}{x}$ (0/0 form if directly computed

$= \underset{x \to 0}{Lt} \dfrac{\left(\sqrt{x+2} - \sqrt{2}\right)\left(\sqrt{x+2} + \sqrt{2}\right)}{x\left(\sqrt{x+2} + \sqrt{2}\right)}$

$= \underset{x \to 0}{Lt} \dfrac{(x+2) - 2}{x\left(\sqrt{x+2} + \sqrt{2}\right)}$

$= \underset{x \to 0}{Lt} \dfrac{1}{\sqrt{x+2} + \sqrt{2}} = \dfrac{1}{2\sqrt{2}}$ //

4. $f(x) = (x^4 + x^3 - 3x^2 + 2x - 1)/(x-1)$

Is $f(x)$ defined at $x = 1$?

Not by direct substitution because it gives 0/0 form.

Could $(x-1)$ be removed as a factor from the numerator?

(P.T.O)

$$
\begin{array}{r|l}
 & x^3 + 2x^2 - x + 3 \\
\hline
(x-1) & x^4 + x^3 - 3x^2 + 2x - 1 \\
 & x^4 - x^3 \\
\hline
 & \quad 2x^3 - 3x^2 + 2x - 1 \\
 & \quad 2x^3 - 2x^2 \\
\hline
 & \qquad -x^2 + 2x - 1 \\
 & \qquad -x^2 - x \\
\hline
 & \qquad\qquad 3x - 1 \\
 & \qquad\qquad 3x \\
\hline
 & \qquad\qquad\quad -1
\end{array}
$$

$\Rightarrow$ It's not a factor.

So strictly speaking $f(0)$ is not defined.

But we could still define it through its limits if the limits exist and are equal.

$$\mathrm{Lt}_{x \to 1^+} f(x) = \mathrm{Lt}_{\epsilon \to 0} f(1+\epsilon) \qquad \epsilon > 0$$

$$= \mathrm{Lt}_{\epsilon \to 0} \frac{(1+\epsilon)^4 + (1+\epsilon)^3 - 3(1+\epsilon)^2 + 2(1+\epsilon) - 1}{((1+\epsilon) - 1)}$$

$$= \mathrm{Lt}_{\epsilon \to 0} \frac{a_4 \epsilon^4 + a_3 \epsilon^3 + a_2 \epsilon^2 + a_1 \epsilon^1 + a_0}{\epsilon}$$

$$= \underset{\epsilon \to 0}{Lt} \left(a_4 \epsilon^3 + a_3 \epsilon^2 + a_2 \epsilon + a_1 + \frac{a_0}{\epsilon} \right)$$

$$= \underset{\epsilon \to 0}{Lt} \left(a_1 + \frac{a_0}{\epsilon} \right) \quad \text{--- (1)}$$

Here, a_i, $i = 0, 1, 2, 3, 4$ are the coefficients emerging from the expansion of the polynomials in the numerator.

$$a_0 = \underset{\text{from } (1+\epsilon)^4}{1} + \underset{\text{from } (1+\epsilon)^3}{1} - \underset{\text{from } -3(1+\epsilon)^2}{3} + \underset{\text{from } 2(1+\epsilon)}{2} - 1 = 0$$

$$a_1 = 4 + 3 - 6 + 2 = 3$$

$$\therefore \underset{\epsilon \to 0}{Lt} f(1+\epsilon) = 3 \quad \text{--- (2)}$$

$$\text{III}^{ly} \underset{\epsilon \to 0}{Lt} f(1-\epsilon) = \underset{\epsilon \to 0}{Lt} \frac{\left[(1-\epsilon)^4 + (1-\epsilon)^3 - 3(1-\epsilon^2) + 2(1-\epsilon) - 1 \right]}{1 - \epsilon - 1}$$

$$= -\underset{\epsilon \to 0}{Lt} \left\{ a_4 \epsilon^3 - a_3 \epsilon^2 + a_2 \epsilon - a_1 + \frac{a_0}{\epsilon} \right\}$$

$$= 3 \quad \text{--- (3)}$$

$$\Rightarrow \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 3$$

$\therefore$ if we define $f(1) = 3$

we can say $f(x)$ is continuous at $x = 1$. (Removable discontinuity)

5. (a) $2x^5 + 2x + 1 = 0 \quad x \in [-1, 1]$

Let $f(x) = 2x^5 + 2x + 1 \rightarrow$ continuous in the interval

$f(-1) = -2 - 2 + 1 = -3 \quad < 0$

$f(1) = 2 + 2 + 1 = 5 \quad > 0$

$\therefore$ By intermediate value theorem, $f(x)$ must attain all values between -3 & 5, including 0.

$\therefore$ There exists at least one point c in $[-1, 1]$ s.t. $f(c) = 0$, i.e. c is a solution to $f(x) = 0$

(b) $\cos x = x$ in $[0, \pi/2]$

Let $f(x) = \cos(x) - x \rightarrow$ continuous in the interval

$$f(0) = \underset{>0}{1}, \quad f(\pi/2) = -\pi/2 < 0$$

$\therefore$ there exist at least one point c in $[0, \pi/2]$ s.t. $f(c) = 0$ &

$$\cos(c) = c.$$

(Applying intermediate value theorem)

6. $f(x) = x^2$. $\quad f(2) = 4$

We have to show that, given an $\epsilon > 0$, however small, we can find a $\delta > 0$, such that

$$|f(x) - 4| < \epsilon \quad — ①$$

$$\text{if } 0 < |x - 2| < \delta \quad — ②$$

Now, $|x^2 - 4| = |x+2||x-2| = |(x-2)+4||x-2|$

$$< \delta(\delta + 4)$$

$$(\because |(x-2)+4| \leq |x-2| + 4 < \delta + 4)$$

For ① to hold we can choose δ such that $\delta(\delta+4) < \epsilon$

$$\text{or} \quad \delta^2 + 4\delta - \epsilon < 0 \quad — ③$$

$$(\delta - \delta_+)(\delta - \delta_-) < 0$$

$$\text{where} \quad \delta_\pm = -2 \pm \sqrt{4+\epsilon} \quad — ④$$

7. $$f(x) = \frac{(x^3-1)}{(x^2-1)} = \frac{(x-1)(x^2+1+x)}{(x-1)(x+1)}$$

$$= \frac{x^2+x+1}{x+1} \quad — ①$$

This does not have a discontinuity at $x=1$. Thus, the original discontinuity is removable.

Using ① $f(1) = 3/2$

So we can assign this value.

8.

at $\frac{\pi}{2}$ $\quad f(x \to \pi/2^-) = f(x \to \pi/2^+)$

$$\Downarrow$$

$$\lim_{\epsilon \to 0} -\sin\left(\frac{\pi}{2} - \epsilon\right) = \lim_{\epsilon \to 0} \left\{ \alpha \sin\left(\frac{\pi}{2} + \epsilon\right) + \beta \right\}$$

i.e. $\quad -\lim_{\epsilon \to 0} \cos\epsilon = \alpha \lim_{\epsilon \to 0} \cos\epsilon + \beta$

$$\Rightarrow \quad \alpha + \beta = -1 \quad — ①$$

at $3\pi/2$ $\quad f(x \to 3\pi/2^-) = f(x \to 3\pi/2^+)$

$$\alpha \lim_{\epsilon \to 0} \sin\left(\frac{3\pi}{2} - \epsilon\right) + \beta = \lim_{\epsilon \to 0} \left(\frac{3\pi}{2} + \epsilon - \frac{3\pi}{2}\right)^2$$

$$-\alpha + \beta = 0$$

or $\alpha = \beta \quad — ②$

$$① \,\&\, ② \Rightarrow \quad \underline{\underline{\alpha = -1/2 = \beta}} \quad — ③$$