CHM 322/642 Assignment 3

March 5, 2017

Due on 20^{th} March, 2016

1. Let $\hat{D}$ be the operator $\frac{d}{dx}$. Verify that $\left(\hat{D}+x\right)\left(\hat{D}-x\right) = \hat{D}^2 - x^2 - 1$.

2. Prove that the product of two linear operators is a linear operator.

3. Consider the infinitesimal translation operator $\hat{T}(dx)$ defined such that $\hat{T}(dx)\Psi(x) = \Psi(x - dx)$

   (a) Show that upto first order in dx we can write $\hat{T}(dx)$ as $\hat{T}(dx) = 1 - i\hat{p}_x/\hbar$, given that $\hat{p}_x\Psi(x) = \frac{\hbar}{i}\frac{d\Psi(x)}{dx}$.

   (b) Show that upto first order in dx, the infinitesimal translation operator obeys the relations (a) $\hat{T}(dx)\hat{T}(dx') = \hat{T}(dx + dx')$, (b) $\hat{T^{-1}}(dx) = \hat{T}(-dx)$, and (c) $\hat{T}$ is unitary.

   (c) For finite translations show that we can write $\hat{T}(h) = exp(-i\frac{\hat{p}_x}{\hbar}h)$ where $T\hat{(h)}\Psi(x) = \Psi(x + h)$.

4. If $\left[\hat{A}, \hat{B}\right] = 0$ then show that $\left[\hat{A}, f(\hat{B})\right] = 0$ for any well-defined function f.

5. Check whether the following operators are Hermitian : (a)$\hat{x}^3$, (b) $\hat{p}_x^4$, (c) $\hat{x}\hat{p}_x$, and (d) $(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})$.

6. The 3-dimensional harmonic oscillator is described by the potential energy function $V(x) = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + \frac{1}{2}k_z z^2$, where k_x, k_y, k_z are force constants. Solve the Schrödinger equation for the system and

determine the energy eigenvalues and eigenfunctions. If all the force constants were the same then what would be the degeneracy of each of the four lowest energy levels.

7. From the definition of the Hermite polynomials verify the recurrence relation : $\rho H_n(\rho) = nH_{n+2}(\rho) + \frac{1}{2}H_{n+1}(\rho)$.

8. For the ground-state of the 1-dimensional harmonic oscillator calculate the expectation values of (a) $\hat{x}$, (b) $\hat{x}^2$, (c) $\hat{p_x}$, and (d) $\hat{p_x}^2$. Find the average value of the kinetic and potential energy and verify that $\langle T \rangle = \langle V \rangle$ in this case.

9. A simple model for the 1-dimensional version of the H-atom is described by the potential $V(x) = \frac{1}{2}k(x - a_0)^2$, where a_0 is the first Bohr radius and x is the distance between the electron and the proton. Write down the potential energy if an electric field $\vec{E} = \epsilon\hat{i}$ is applied on the atom and solve the system for eigenfunctions and eigenvalues.

10. Find the eigenvalues and eigenfunctions of $\hat{H}$ for a 1-dimensional system with $V(x) = \infty$ for $x < 0$, $V(x) = \frac{1}{2}kx^2$ for $x \geq 0$.