General Relativity - 2017

Indian Institute of Science Education and Research Bhopal

Assignment 3

1. Write down the equations of motion of a particle (in polar coordinates) moving in a central force. From these equations, show how the gravitational force term can be reinterpreted in terms of a curved geometry without any gravitational force. Try to guess a metric (not necessarily a solution of the Einstein's equation) that can provide similar geodesic equations.

2. When a vector V_a is parallel transported along a geodesics, show that the angle subtended by the vector and the geodesics (i.e the tangent of the geodesics) is unchanged.

3. For a diagonal metric, prove that the Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\lambda} = 0, \quad \Gamma^{\mu}_{\lambda\lambda} = -\frac{1}{2g_{\mu\nu}}\frac{\partial g_{\lambda\lambda}}{\partial x^{\mu}}, \quad \Gamma^{\mu}_{\mu\lambda} = \frac{\partial}{\partial x^{\lambda}}\left(\ln(\sqrt{|g_{\mu\mu}|})\right), \quad \gamma^{\mu}_{\mu\mu} = \frac{\partial}{\partial x^{\mu}}\left(\ln(\sqrt{|g_{\mu\mu}|})\right)$$

4. (a) Let u^b be the 4-velocity of a particle and a^b its acceleration 4-vector. Show that $u^b \nabla_a u_b = 0$ and $u^b a_b = 0$.
(b) Show that the tensor defined by $F_{ab} = \nabla_a A_b - \nabla_b A_a$ can also be written as $F_{ab} = \partial_a A_b - \partial_b A_a$, *i.e* that the Christoffel symbols drop out.
(c) The three dimensional curl of a vector $\vec{B}$ is defined by $(\vec{\nabla} \times \vec{B})^i = \epsilon^{ijk} B_k$. Show that this can also be written as $(\vec{\nabla} \times \vec{B})^i = \epsilon^{ijk}\partial_j B_k$.

5. The flat spatial metric in spherical coordinates is given by $ds^2 = dr^2 + r^2 d\Omega^2$ where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$. Write out the Laplacian $\nabla^2 f = g^{ij}\nabla_i \nabla_j f = \nabla_i \nabla^j f$ in terms of the partial derivatives acting on f.

6. Show explicitly that $\nabla_a g_{bc} = 0$.

7. Show that with the following transformation $x'^a = x^a - \frac{1}{2}\Gamma^a_{bc}|_o x^b x^c$, you can always make Γ^a_{bc} to be zero locally.

8. Compute all the non vanishing components of the Riemann tensor R_{ijkl} $(i, j, k, l = \theta, \phi)$ for the 2-sphere metric where a is the radius of the sphere. Show that the Ricci tensor is given by $R_{ij} = g_{ij}/a^2$. Find also the scalar curvature.

9. Show that $D_a F^{ab} = \frac{1}{\sqrt{-g}}\partial a(\sqrt{-g}F^{ab})$

10. If the Riemann curvature is isotropic, the Riemann curvature tensor can be written as

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

Use the Bianchi identities to show that K must be a constant. (Schur's theorem). Find Ricci scalar in this case.

11. Show that the Ricci tensor is a symmetric tensor.