General Relativity - 2017

Indian Institute of Science Education and Research Bhopal

Assignment 1

Welcome to some muscle flexing ..

1. Consider two inertial frames S and S' moving with a relative velocity v along the x-direction. Find out acceleration of a particle in the S' frame in terms of acceleration in S frame.

These transformation formulae should tell you that acceleration is not *invariant* in special relativity, unlike Newtonian mechanics. But, the formula also reveal that acceleration is an *absolute* quantity, that is, all observers agree upon whether a body is accelerating. If the acceleration is zero in one inertial frame, it is necessarily zero in any other frame. As we will learn soon, this is no longer going to be true in General Relativity.

2. According to Newtonian mechanics, a particle under constant force will eventually have infinite velocity. Just to make sure, convince yourself, and draw the space-time diagram for this classical particle.
Obviously, this is not correct in the 'true' Special Theory of Relativity! Find out the trajectory $x(\tau), t(\tau)$ of a particle A with constant acceleration f. Draw the world-line in this case assuming initial condition $x(t=0)=0$. Asymptotes of the world-line is called the 'horizon' of the space-time. We will come back to this issue later!
In the same space-time digram, draw the worldliness of a rest particle B at $x=0$. Show that no message sent by B after $t=c/f$ will ever reach A. This signifies the term 'horizon' for the particle.
Check the norm of the four-velocity, and also show explicitly that the particle's three-velocity never exceeds the speed of light. Calculate the norm of the four-acceleration.

3. (a)Transform the line element of special relativity from the usual (t,x,y,z) to a new coordinate system (t',x',y',z') related by

$$\begin{aligned} t &= \left(\frac{c}{g}+\frac{x'}{c}\right)\ Sinh\left(\frac{gt'}{c}\right) \\ x &= c\left(\frac{c}{g}+\frac{x'}{c}\right)\ Cosh\left(\frac{gt'}{c}\right)-\frac{c^2}{g} \\ y &= y' \\ z &= z', \end{aligned}$$

for constant g with the dimension of acceleration.

(b) For $gt'/c \ll 1$, show that this corresponds to a transformation to a uniformly accelerated frame in Newtonian mechanics.

(c) Show that a clock at rest in this frame $x' = h$ runs fast compared to a clock at rest at $x' = 0$ by a factor $(a + gh/c^2)$.

4. 3-velocity corresponding to the 4-velocity $\mathbf{u}$ is $\vec{u}$. Express (a) u^0 in terms of $|\vec{v}|$, (b) $u^j (j = 1, 2, 3)$ in term of $\vec{v}$, (c) u^0 in terms of u^j (d) $\mathrm{d}/\mathrm{d}\tau$ in terms of d/dt and $\vec{v}$, (e) v^j in terms of u^j, and (f) $\vec{v}$ in terms of u^0.

5. Consider a particle moving along the x-axis whose velocity as a function of time is

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1 + g^2t^2}},$$

where g is a constant.

(a) Does the particle's speed ever exceed the speed of light? (b) Calculate the components of the particle's four-velocity. (c) Express x and t as a function of the proper time along the trajectory. (d) What are the components of the four-force and the three-force acting on the particle?

6. Work out the components of the four-acceleration vector $\mathbf{a} = \frac{\mathbf{du}}{\mathbf{d}\tau}$ in terms of three velocity $\vec{V}$ and the three-acceleration $\vec{a} = \frac{d\vec{V}}{dt}$. Now verify explicitly that $\mathbf{a} \cdot \mathbf{u} = \mathbf{0}$.

7. (a) An accelerated lab has a bottom at $x' = 0$ and a top at $x' = h$. Use the line-element derived in the last problem to show that the heigh of the lab remains constant in time, *i.e* the lab moves rigidly.

(b) Compute the invariant acceleration $a = (\mathbf{a} \cdot \mathbf{a})^{\mathbf{1/2}}$ where $a^\alpha = \frac{d^2x^\alpha}{d\tau^2}$, and show that it is different for the top and bottom of the laboratory.

8. Show explicitly that the electromagnetic wave equation is invariant under Lorentz transformation.

9. We would like to construct the Lorentz transformation matrix for a most generic boost transformation $\vec{\beta}$. The boost vector $\vec{\beta}$ has components β_i where, i runs over 1, 2 and 3. The boost vector $\vec{\beta}$ is $SO(3)$ vector.

Break the Lorentz transformation matrix Λ^a_b in terms of $SO(3)$ scalars, vectors and tensors and use the property

$$\eta_{cd}\Lambda^c_a\Lambda^d_b = \eta_{ab}$$

to find all the compponents of Λ^a_b what we obtained in the class.

10. Define a four vector $A^a = \{\frac{\phi}{c}, \vec{A}\}$, where ϕ is scalar potential and $\vec{A}$ is vector or magnetic potential. Show that the four sets of Maxwell's equations can be obtained from the following two sets of equations.

$$\partial_a F^{ab} = \mu_0 J^b \qquad \text{and} \qquad \partial_a G^{ab} = 0$$

where,

$$F_{ab} = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b}, \qquad \text{and} \qquad G^{ab} = \frac{1}{2}\epsilon^{abcd}F_{cd}$$

and J^a is given by,

$$J^a = \rho_0 u^a, \;\; \rho_0 \text{ is the rest charge density.}$$

Identify different components of F_{ab} and G_{ab} with different components of the electric and magnetic fields such that you obtain the correct Maxwell's equations from the above two equation given.

11. A coordinate system K' moves with a velocity $\vec{v}$ relative to another system K. In K', a particle have velocity $\vec{u}'$ and an acceleration $\vec{a}'$. Find the Lorentz transformation law for accelerations, and show that in the system K the components of acceleration parallel and perpendicular to $\vec{v}$ are

$$\begin{aligned}
\vec{a}_{||} &= \frac{(1 - v^2/c^2)^{3/2}}{(1 + \vec{v}.\vec{u'}/c^2)^3}\vec{a'_{||}} \\
\vec{a}_{\perp} &= \frac{(1 - v^2/c^2)}{(1 + \vec{v}.\vec{u'}/c^2)^3}(\vec{a}'_{\perp} + \frac{\vec{v}}{c^2} \times (\vec{a}' \times \vec{u}')
\end{aligned}$$