Tutorial 2

Vector Analysis

PHY 102

January 16, 2015

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• Given
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
, show that, a) $\vec{\nabla}(\ln |\vec{r}|) = \frac{\hat{r}}{r}$, b)
 $\vec{\nabla}(\frac{1}{r}) = -\frac{\hat{r}}{r^2}$ and c) $\vec{\nabla}r^n = nr^{n-1}\vec{r}$ where
 $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

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• Recall the gradient of a scalar Φ :

$$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}$$

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$$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}$$

• Here $\Phi = \ln |\vec{r}| = \ln \sqrt{x^2 + y^2 + z^2} = \frac{1}{2} \ln(x^2 + y^2 + z^2).$

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• Show that
$$\vec{\nabla}^2\left(\frac{1}{r}\right) = 0.$$

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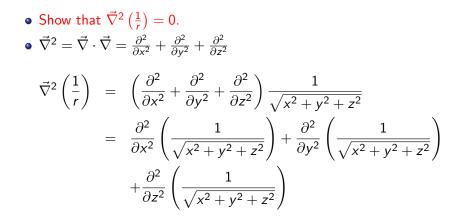
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 $\vec{\nabla}^2 \left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

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• Show that
$$\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = 0$$
, for $r \neq 0$ where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

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- Divergence of a vector is given by,

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

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Here,
$$V_x = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$
, $V_y = \frac{y}{(x^2+y^2+z^2)^{3/2}}$ and $V_z = \frac{z}{(x^2+y^2+z^2)^{3/2}}$

- Show that $\vec{\nabla} f(x, y, z)$ is a vector perpendicular to the surface f(x, y, z) = constant.
- Let f(x, y, z) be a function of 3 variables. f(x, y, z) = c defines a surface in three dimensions.
- This means, any point on the surface must satisfy this equation.
- Suppose *P* is any point on that surface.
- We calculate \$\vec{\nabla}\$f\$ at \$P\$. Goal is to show that \$\vec{\nabla}\$f\$ is perpendicular to the surface at the point \$P\$.

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- This means, any point on the surface must satisfy this equation.
- Suppose *P* is any point on that surface.
- We calculate $\vec{\nabla}f$ at *P*. Goal is to show that $\vec{\nabla}f$ is perpendicular to the surface at the point *P*.

- That means we need to show ∇f is perpendicular to the tangent to any curve that lies on the surface and goes through P.
- We consider a curve which lies on the surface and passing through *P*.
- How can we specify a curve on the surface?
- Think of a particle moving on the surface.
- Therefore, its trajectory is a curve on the surface.
- Let $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is the position vector of the particle. x(t), y(t), z(t) are the coordinates of the particle at time t.
- Since the particle lies on the surface: f(x(t), y(t), z(t)) = c.

- That means we need to show $\vec{\nabla} f$ is perpendicular to the tangent to any curve that lies on the surface and goes through *P*.
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• What is the tangent to this curve ?

- Ans: We know velocity of a particle is a tangent to the trajectory.
- Therefore, $\vec{T} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$ is the tangent vector to the curve at time t.
- Since f(x(t), y(t), z(t)) = c. Differentiate both sides with t.
 We get,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$
$$=> \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) = 0$$
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