Tutorial 2

Vector Analysis

PHY 102

January 16, 2015

Problem no. 2

- Given $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that, a) $\vec{\nabla}(\ln|\vec{r}|) = \frac{\hat{r}}{r}$, b) $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$ and c) $\vec{\nabla} r^n = n r^{n-1}\vec{r}$ where $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

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- Recall the gradient of a scalar Φ:

$$\vec{\nabla}\Phi = \frac{\partial \Phi}{\partial x}\hat{i} + \frac{\partial \Phi}{\partial y}\hat{j} + \frac{\partial \Phi}{\partial z}\hat{k}$$

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$$\vec{\nabla}\Phi = \frac{\partial \Phi}{\partial x}\hat{i} + \frac{\partial \Phi}{\partial y}\hat{j} + \frac{\partial \Phi}{\partial z}\hat{k}$$

- Here $\Phi = \ln|\vec{r}| = \ln\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\ln(x^2 + y^2 + z^2)$.

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$$\vec{\nabla}^2 \left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

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$$\begin{aligned}
\vec{\nabla}^2 \left(\frac{1}{r}\right) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \frac{1}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) \\
&\quad + \frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)
\end{aligned}$$

Problem no. 9

- Show that $\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = 0$, for $r \neq 0$ where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

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Here, $V_x = \frac{x}{(x^2+y^2+z^2)^{3/2}}$, $V_y = \frac{y}{(x^2+y^2+z^2)^{3/2}}$ and $V_z = \frac{z}{(x^2+y^2+z^2)^{3/2}}$

Problem no. 10

- Show that $\vec{\nabla} f(x, y, z)$ is a vector perpendicular to the surface $f(x, y, z) = constant$.
- Let $f(x, y, z)$ be a function of 3 variables. $f(x, y, z) = c$ defines a surface in three dimensions.
- This means, any point on the surface must satisfy this equation.
- Suppose P is any point on that surface.
- We calculate $\vec{\nabla} f$ at P. Goal is to show that $\vec{\nabla} f$ is perpendicular to the surface at the point P.

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- That means we need to show $\vec{\nabla} f$ is perpendicular to the tangent to any curve that lies on the surface and goes through P.
- We consider a curve which lies on the surface and passing through P.
- How can we specify a curve on the surface?
- Think of a particle moving on the surface.
- Therefore, its trajectory is a curve on the surface.
- Let $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is the position vector of the particle. $x(t), y(t), z(t)$ are the coordinates of the particle at time t.
- Since the particle lies on the surface: $f(x(t), y(t), z(t)) = c$.

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- What is the tangent to this curve ?
- Ans: We know velocity of a particle is a tangent to the trajectory.
- Therefore, $\vec{T} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$ is the tangent vector to the curve at time t.
- Since $f(x(t), y(t), z(t)) = c$. Differentiate both sides with t. We get,

$$\begin{aligned} \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} &= 0 \\ => \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) &= 0 \\ \vec{\nabla} f \cdot \vec{T} &= 0 \end{aligned}$$

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