Solution - Mid Sem

Q.1. $\vec{V}_1 = 3\hat{i} - \hat{j} + 2\hat{k}$

$\vec{V}_2 = -2\hat{i} + 5\hat{j} - 4\hat{k}$

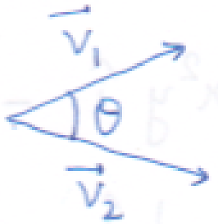
i) $\vec{V}_1 \cdot \vec{V}_2 = -6 - 5 - 8 = -19.$

ii) $$\vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ -2 & 5 & -4 \end{vmatrix}$$

$$= \hat{i}(4-10) - \hat{j}(-12+4) + \hat{k}(15-2)$$

$$= -6\hat{i} + 8\hat{j} + 13\hat{k}$$

iii) $\vec{V}_1 \cdot \vec{V}_2 = |\vec{V}_1||\vec{V}_2| \cos\theta$



$$-19 = \sqrt{9+1+4}\,\sqrt{4+25+16}\,\cos\theta$$

$$= \sqrt{2}\sqrt{7}\,\sqrt{5}\,3\cos\theta$$

$$\Rightarrow \cos\theta = -\frac{19}{3\sqrt{70}}$$

$$\boxed{\theta = \cos^{-1}\left[-\frac{19}{3\sqrt{70}}\right]}$$

(iv) Projection of $\vec{V}_1$ along $\hat{i} + \hat{j} - \hat{k}$

$$= \vec{V}_1 \cdot \frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}}(3\hat{i} - \hat{j} + 2\hat{k})(\hat{i} + \hat{j} - \hat{k})$$

$$= \frac{1}{\sqrt{3}}(3 - 1 - 2) = 0$$

2) $\vec{V} = 2x\,\hat{i} + 5y^2\,\hat{j} + 3e^2\,\hat{k}$

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(2x\,\hat{i} + 5y^2\,\hat{j} + 3e^2\,\hat{k}\right)$$

$$= 2 + 10y + 3e^2$$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 5y^2 & 3e^2 \end{vmatrix} = 0$$

3. $\vec{P} = x^2y\,\hat{i} - 2xy\,\hat{j} + 3(x^2+y^2)\,\hat{k}$

$$\vec{\nabla}\times\vec{P} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xy & 3(x^2+y^2) \end{vmatrix}$$

$$= \hat{i}\,(6y - 0) - \hat{j}\,(6x - 0) + \hat{k}\,(-2y - x^2)$$

$$= 6y\,\hat{i} - 6x\,\hat{j} - (2y + x^2)\,\hat{k}$$

$d\vec{a} = \hat{k}\,dx\,dy.$

P.T.O

LHS

$$\int_A (\vec{\nabla}\times\vec{P})\cdot d\vec{a}$$

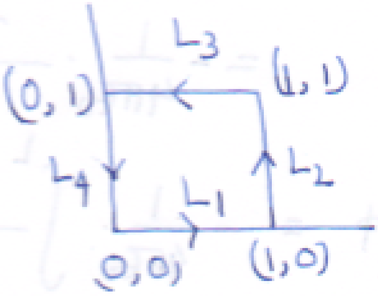
$$= -\int_0^1 dy \int_0^1 dx \left(+2y + x^2\right)$$

$$= -\int_0^1 dx \int_0^1 (+2)\, y\, dy - \int_0^1 dy \int_0^1 x^2\, dx$$

$$\boxed{\int_A (\vec{\nabla}\times\vec{P})\cdot d\vec{a} = -1 - \frac{1}{3} = -\frac{2}{3}}$$

R.H.S.

$$\oint_L \vec{P}\cdot d\vec{l}$$



$$= \int_{L_1} \vec{P}\cdot d\vec{l} + \int_{L_2} \vec{P}\cdot d\vec{l} + \int_{L_3} \vec{P}\cdot d\vec{l} + \int_{L_4} \vec{P}\cdot d\vec{l}$$

on L_1: $dy = dz = 0$, on L_2: $dx = dz = 0$ on L_3: $dy = dz = 0$
on L_4: $dx = dz = 0$

$$\int_{L_1} \vec{P}\cdot d\vec{l} = y\int_0^1 dx\, x^2 = 0 \quad \because y = 0 \text{ on } L_1$$

$$\int_{L_2} \vec{P}\cdot d\vec{l} = -2x\int_0^1 y\, dy = -1 \quad \because x = 1 \text{ on } L_2$$

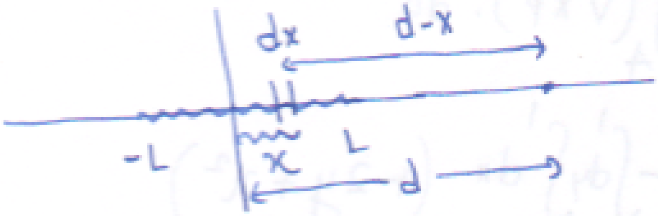
$$\int_{L_3} \vec{P}\cdot d\vec{l} = y\int_1^0 x^2\, dx = -\frac{1}{3} \quad \because y = 1 \text{ on } L_3$$

$$\int_{L_4} \vec{P}\cdot d\vec{l} = -2x\int_1^0 y\, dy = 0 \quad \because x = 0 \text{ on } L_4$$

$$\therefore \boxed{\oint_L \vec{P}\cdot d\vec{l} = -1 - \frac{1}{3} = -\frac{2}{3}}$$

4. For 4a look at class note.

4b



Potential at P due to ~~a~~ infinitesimal line charge dx from the origin

$$\boxed{d\phi = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{d-x}}$$

$$= -\frac{1}{4\pi\epsilon_0}\cdot\frac{x^2 dx}{x-d}$$

$$\phi = -\frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\frac{x^2 dx}{x-d}$$

$$= -\frac{1}{4\pi\epsilon_0}\int_{-L}^{L}\left[\frac{x^2-d^2}{x-d}dx + \frac{d^2 dx}{x-d}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\int_{-L}^{L}(x+d)\,dx - \frac{d^2}{4\pi\epsilon_0}\int_{-L}^{L}\frac{dx}{x-d}$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\frac{1}{2}x^2 + dx\right|_{-L}^{L} - \frac{d^2}{4\pi\epsilon_0}\cdot \ln(x-d)\Big|_{-L}^{L}$$

$$= -\frac{1}{4\pi\epsilon_0}d(L+L) - \frac{d^2}{4\pi\epsilon_0}\cdot\ln\frac{L-d}{-L-d}$$

$$\boxed{\phi = \frac{dL}{2\pi\epsilon_0} + \frac{d^2}{4\pi\epsilon_0}\cdot\ln\frac{L+d}{d-L}}$$

Therefore potential at any point y from the origin is given by

$$\phi(y) = \frac{yL}{2\pi\epsilon_0} + \frac{y^2}{4\pi\epsilon_0} \ln \frac{L+y}{L-y}$$

Therefore Electric field is given by

$$\vec{E} = - \cancel{\nabla} \frac{\partial \phi}{\partial y} \hat{i}$$

$$= - \frac{L}{2\pi\epsilon_0} \hat{i} - \frac{2y}{4\pi\epsilon_0} \ln \frac{L+y}{L-y} \hat{i}$$

$$- \frac{y^2}{4\pi\epsilon_0} \cdot \frac{(L-y)}{L+y} \frac{(L-y)+(L+y)}{(L-y)^2} \hat{i}$$

putting $y = d$

$$\vec{E} = - \frac{L}{2\pi\epsilon_0} \hat{i} - \frac{2d}{4\pi\epsilon_0} \ln \frac{L+d}{L-d} \hat{i}$$

$$- \frac{d^2}{4\pi\epsilon_0} \frac{2L}{L^2 - d^2} \hat{i}$$

The same electric field can also be found directly applying coulombs law

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{-L}^{L} \frac{x^2 dx}{(d-x)^2} \hat{i}$$

(5) Look at assignment (5).