The Dirac Equation

By analogy with non-relativistic Schrödinger Eqns Dirac started of the following eqn.

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r},t) = H\,\psi(\vec{r},t)$$

In Schrödinger Eqn. $H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r})$.

But this eqn. is not Lorentz covariant then, because ~~there is~~ the eqn. is linear in $\frac{\partial}{\partial t}$ and quadratic in $\frac{\partial}{\partial x^i}$.

As we know in a relativistic theory space coordinates and time coordinate must enter on the same footing. Therefore, Dirac considered the Hamiltonian is also linear in $\partial/\partial x^i = p^i$

$$H = c\,\vec{\alpha}\cdot\hat{\vec{p}} + \beta mc^2$$

$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are constant and same as β.

$$H = c\sum_{i=1}^{3} \alpha_i \hat{p}_i + \beta mc^2 \qquad \hat{\vec{p}} = -i\hbar\vec{\nabla}$$

Therefore we have

$$\boxed{i\hbar \frac{\partial}{\partial t}\psi = -i\hbar c\,\vec{\alpha}\cdot\vec{\nabla}\psi + \beta mc^2\psi}$$

Eqn. for a free particle.

Since the energy operator $\hat{E} = i\hbar\frac{\partial}{\partial t}$

$$\left(\hat{E} - c\vec{\alpha}\cdot\hat{\vec{p}} - \beta mc^2\right)\psi = 0 \qquad (1)$$

Since ψ is wave func^n for a relativistic particle it must satisfy the corresponding relativistic energy-momentum ~~eq~~ eq^n.

$$\left(\hat{E}_{op}^2 - \hat{p}^2c^2 - m^2c^4\right)\psi = 0$$

This requirement is based on the correspondence principle. It guarantees that in the classical limit the free wave packet sol^n of the Dirac eq^n will exibit correct relativistic energy momentum relation for free particle.

Multiply (1) by $\left(\hat{E} + c\vec{\alpha}\cdot\hat{\vec{p}} + \beta mc^2\right)$ on the left

$$\left\{\hat{E}^2 - c^2\sum_{i=1}^{3}\alpha_i\hat{p}_i\sum_{j=1}^{3}\alpha_j\hat{p}_j - m^2c^4\beta^2 - c\sum\left(\alpha_i\hat{p}_i\beta + \beta\alpha_i\hat{p}_i\right)\right\}\psi = 0$$

$$\Rightarrow \left\{\hat{E}^2 - c^2\left(\sum_{i=1}^{3}\alpha_i^2\hat{p}_i^2 + \sum_{i\neq j}\left(\alpha_i\alpha_j + \alpha_j\alpha_i\right)\hat{p}_i\hat{p}_j\right) - c\sum_{i=1}^{3}\left(\alpha_i\beta + \beta\alpha_i\right)\hat{p}_i - m^2c^4\beta^2\right\}\psi = 0$$

$\Rightarrow \quad \alpha_i^2 = 1 \quad$ for $i = 1, 2, 3$

$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad \forall \; i \neq j$

$\alpha_i \beta + \beta \alpha_i = 0 \quad \forall \; i$

$\beta^2 = 1$

Any ordinary number can can not satisfy these relations.

- α_i and β are matrices. Let us consider $(n \times n)$ mat.

- Since $\hat{H}$ is hermitian $\hat{H} = \hat{H}^\dagger \Rightarrow \alpha_i = \alpha_i^\dagger$, $\beta = \beta^\dagger$.

If $\vec{\alpha}$ and β are $(n \times n)$ matrices then Dirac eqn. is meaningful only when ψ is a column vector.

First observe that $\alpha_i^2 = 1$, $\beta^2 = 0 \Rightarrow$ Eigenvalues of α_i and β also satisfy same eqn. Thus α_i and β has eigenvalues ± 1.

$\alpha_k \beta = -\beta \alpha_k$

$\Rightarrow \alpha_k = -\beta \alpha_k \beta$

$\Rightarrow tr(\alpha_k) = -tr(\beta \alpha_k \beta) = -tr\, \alpha_k$

$\Rightarrow \boxed{tr\, \alpha_k = 0}$ $\quad$ Ily $tr(\beta) = 0$

$\Rightarrow$ $\boxed{\alpha_i \text{ and } \beta \text{ are even dimensional}}$

So n can be 2.

It is not possible two find 4 (2×2) matrices which anticommute of eachother. There are maximum 3 2×2 matrices which anticommute of each other: $\sigma_x, \sigma_y, \sigma_z$.

Therefore minimum value for n is 4.

We choose a particular representation — Dirac representation.

$$\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 𝟙 & 0 \\ 0 & -𝟙 \end{pmatrix}$$

Ex: 26: Show that all the ~~anticomm~~ properties of $\vec{\alpha}$ and β are satisfied.

Hence ψ is a 4 dimensional column vector

$$\psi(\vec{x},t) = \begin{pmatrix} \psi_1(\vec{x},t) \\ \psi_2(\vec{x},t) \\ \psi_3(\vec{x},t) \\ \psi_4(\vec{x},t) \end{pmatrix} \longrightarrow \text{Spinor field.}$$

~~This 4 has no Dimension of $\vec{\alpha}$ and β has no connection is ...~~

ψ represents a spin 1/2 particle.

$$i\hbar \frac{\partial}{\partial t}\psi + i\hbar c\, \alpha_i \partial_i \psi - \beta m c^2 \psi = 0$$

$$\Rightarrow \left(i \frac{\beta}{c}\frac{\partial}{\partial t} + i \beta \alpha_i \partial_i - \frac{mc}{\hbar}\right)\psi = 0 \quad \left\{ \text{muliply } \frac{1}{\hbar c}\beta \right)$$

Define $\gamma_i = -\beta\alpha_i$

$\gamma_0 = \beta$

Hence $\gamma^i = \beta\alpha_i$, $\gamma^0 = \beta$

$$\left(i\gamma^0 \partial_0 + i\gamma^i \partial_i - \frac{mc}{\hbar}\right)\psi = 0$$

$$\gamma^\mu = \{\gamma^0, \gamma^i\}$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \hbar = c = 1$$

$$\boxed{(i\not{\partial} - m)\psi = 0} \qquad \not{\partial} = \gamma^\mu \partial_\mu$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^i = \beta\alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

Ex 27: Show that

$$\gamma^{\mu\dagger} = \gamma^\mu \quad \text{and} \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$$

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \tilde{\psi}(\vec{p})$$

$$\psi(\vec{x},t) = e^{-ip_0 t} \psi(\vec{x})$$

$$= \int \frac{d^3p}{(2\pi)^3} \cdot e^{-ip\cdot x} \tilde{\psi}(\vec{p})$$

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \qquad x = \{\vec{x}, t\}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot x} \tilde{\psi}(\vec{p})$$

$$i\gamma^\mu \partial_\mu \psi = \int \left[(-i)\, i\, \gamma^\mu p_\mu \; \cancel{m}\right] e^{-ip\cdot x} \tilde{\psi}(\vec{p}) \frac{d^3p}{(2\pi)^3}$$

$$\Rightarrow (i\gamma^\mu \partial_\mu - m)\psi = \int (\gamma^\mu p_\mu - m) e^{-ip\cdot x} \tilde{\psi}(p) \frac{d^3p}{(2\pi)^3} = 0$$

$$\Rightarrow (\gamma^\mu p_\mu - m)\tilde{\psi}(\vec{p}) = 0$$

$$\Rightarrow \begin{pmatrix} p_0 - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(p_0+m) \end{pmatrix} \begin{pmatrix} \tilde{\varphi}(\vec{p}) \\ \tilde{\chi}(\vec{p}) \end{pmatrix} = 0 \qquad \tilde{\psi}(\vec{p}) = \begin{pmatrix} \tilde{\varphi}(\vec{p}) \\ \tilde{\chi}(\vec{p}) \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} (p_0 - m)\tilde{\varphi}(\vec{p}) - (\vec{\sigma}\cdot\vec{p})\tilde{\chi}(\vec{p}) &= 0 \\ (\vec{\sigma}\cdot\vec{p})\tilde{\varphi}(\vec{p}) - (p_0+m)\tilde{\chi}(\vec{p}) &= 0 \end{aligned}\right\}$$

$$\begin{vmatrix} p_0 - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(p_0 - m) \end{vmatrix} = 0$$

$$\left[(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = (\vec{A}\cdot\vec{B})\mathbb{1} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}) \right]$$

$$\Rightarrow -p_0^2 + m^2 + \vec{p}^2 = 0 \Rightarrow p_0 = \pm\sqrt{\vec{p}^2 + m^2} = \lambda\omega_{\vec{p}} \qquad \lambda = \pm 1$$

Corresponds to '+ve' and '-ve' energy sol^n

$$\tilde{\varphi}(\vec{p}) = \frac{\vec{\sigma}\cdot\vec{p}}{\lambda\omega_{\vec{p}} - m}\,\tilde{\chi}(\vec{p})$$

$$\tilde{\chi}(\vec{p}) = \frac{\vec{\sigma}\cdot\vec{p}}{\lambda\omega_{\vec{p}} + m}\,\tilde{\varphi}(\vec{p})$$

$\tilde{\varphi}(p)$ and $\tilde{\chi}(\vec{p})$ are two component spinors. There are two linearly independent basis

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence $\tilde{\varphi}$ and $\tilde{\chi}$ can be linear combination of these two spinors.

Therefore for a given λ there exists two possible solutions. Hence, we have 4 linearly independent solutions

$\lambda = +1$ We have two solutions

$$u(\vec{p},1) = \mathcal{N}\begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}}{\omega_{\vec{p}}+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$u(\vec{p},2) = \mathcal{N}\begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{\omega_{\vec{p}}+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

For $\lambda = -1$

$$v(-\vec{p},1) = N \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{\omega_p + m}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0 \end{pmatrix}$$

$$v(-\vec{p},2) = N \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{\omega_p + m}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1 \end{pmatrix}$$

N is normalization.

Define: $\bar{u} = u^{\dagger}\gamma^0$

Fix the normalization such that $\bar{u}(\vec{p},1)\, u(\vec{p},1) = 1$

Ex: 28 Show that $|N|^2 = \left(\frac{\omega_p + m}{2m}\right)$

Ex: 29
$$\bar{u}(\vec{p},r)\, u(\vec{p},s) = \delta_{rs}$$
$$\bar{v}(\vec{p},r)\, v(\vec{p},s) = -\delta_{rs}$$
$$u^{\dagger}(\vec{p},r)\, u(\vec{p},s) = \frac{\omega_p}{m}\delta_{rs} = v^{\dagger}(\vec{p},r)\, v(\vec{p},s)$$
$$\bar{v}(\vec{p},r)\, u(\vec{p},s) = 0, \quad \bar{u}(\vec{p},r)\, v(\vec{p},s) = 0$$
$$\sum_s u(\vec{p},s)\, \bar{u}(\vec{p},s) = \frac{\not{p}+m}{2m}$$
$$\sum_s v(\vec{p},s)\, \bar{v}(\vec{p},s) = \frac{\not{p}-m}{2m}$$

Quantization

$$L = \bar{\psi}(x)(i\not{\partial} - m)\psi(x)$$

$$= \psi^{\dagger}_{\alpha}(x)\gamma^{0}_{\alpha\beta}(i\not{\partial}-m)_{\beta\gamma}\psi_{\gamma}$$

$\psi^{\dagger}_{\alpha}$ and ψ_{α} are classical fields

Momentum conjugate to ψ_{α} $\Rightarrow$ $\Pi_{\alpha} = i\psi^{\dagger}_{\alpha}$

To quantize the system:

1. ψ_{α} and Π_{α} are operators in H.S.

2. Quantization conditions: $\{\psi_{\alpha}(\vec{x},t), \Pi_{\beta}(\vec{y},t)\} = i\hbar\delta^{3}(\vec{x}-\vec{y})\delta_{\alpha\beta}$

$$\Rightarrow \{\psi_{\alpha}(\vec{x},t), \psi^{\dagger}_{\beta}(\vec{y},t)\} = \hbar\delta^{3}(\vec{x}-\vec{y})\delta_{\alpha\beta}$$

and $\{\psi_{\alpha}(\vec{x},t), \psi_{\beta}(\vec{y},t)\} = 0$ and $\{\psi^{\dagger}_{\alpha}(\vec{x},t), \psi^{\dagger}_{\beta}(\vec{y},t)\} = 0$

This means in the classical limit i.e. $\hbar \to 0$ ψ_{α} and $\psi^{\dagger}_{\alpha}$ are not ordinary numbers:

$$\psi_{\alpha}(\vec{x},t)\psi_{\beta}(\vec{y},t) = -\psi_{\alpha}(\vec{y},t)\psi_{\beta}(\vec{x},t) \quad \text{etc.}$$

This fact is very important when we consider path integral.

In path integral we are interested to calculate the correlation func^ns

$$\langle \Omega | T\{ \psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_m}(x_m)\, \psi^{+}_{\beta_1}(y_1) \cdots \psi^{+}_{\beta_n}(y_n) \} | \Omega \rangle$$

Here we need to ~~be~~ define the time ordering carefully.

→ The time order product picks up one '−' sign for each interchange of operators that is necessary to put the fields in time order

Consider: $T\{ \psi_1 \psi_2 \psi_3 \psi_4 \}$ and $t_3 > t_4 > t_1 > t_2$

Hence $T\{ \psi_1 \psi_2 \psi_3 \psi_4 \} = (-1)^{2+2} \psi_3 \psi_4 \psi_1 \psi_2$

Path integral prescription

$$\langle \psi_{\alpha_1}(x_1)\, \psi_{\alpha_2}(x_2) \cdots \psi_{\alpha_M}(x_m)\, \psi^{+}_{\beta_1}(y_1) \cdots \psi^{+}_{\beta_n}(x_n) \rangle$$

$$= \frac{\int [D\psi][D\psi^{+}]\, e^{iS} \prod_{l=1}^{m} \psi_{\alpha_l}(x_l) \prod_{k=1}^{n} \psi^{+}_{\beta_k}(y_k)}{\int [D\psi]\, \cancel{e^{iS}}\, [D\psi^{+}]\, e^{iS}}$$