Vacuum expectation value

$$K(q',t';q'',t''; \hat{q}(t_1),\dots \hat{q}(t_n))$$

$$= \sum_{m,n} \langle q''| e^{-i\hat{H}t''}|n\rangle\langle n|T\left\{\prod_{k=1}^{n}\hat{q}(t_k)\right\}|m\rangle\langle m| e^{i\hat{H}t'}|q'\rangle$$

First we want to calculate the matrix element for $t'\to-\infty$ and $t''\to\infty$
Rotate time axis by an infinitesimal imaginary amount

$$t'' = T(1-i\epsilon) \quad \text{and} \quad t' = -T(1-i\epsilon) \quad \text{and take } T\to\infty$$

$$K = \sum_{m,n} e^{-iE_nT(1-i\epsilon)} \langle q''|n\rangle\langle n|T\left\{\prod_{k=1}^{n}\hat{q}(t_k)\right\}|m\rangle\langle m|q'\rangle e^{-iE_m(T-i\epsilon)}$$

$$= \sum_{m,n} e^{-i(E_n+E_m)T} e^{-\epsilon(E_n+E_m)T}$$

$$\langle q''|n\rangle\langle n|T\left\{\prod_{k=1}^{n}\hat{q}(t_k)\right\}|m\rangle \langle m|q'\rangle$$

Now let us take $T\to\infty$ limit.
Since $E_n, E_m > E_0$ $\forall$ $n,m\neq 0$, therefore only ground state will contribute to this sum.

$$K = \lim_{T\to\infty} e^{-2iE_\Omega T} e^{-2\epsilon E_\Omega T} \langle\Omega|T\left\{\prod_{k=1}^{n}\hat{q}(t_k)\right\}|\Omega\rangle$$

$$\times \langle q''|\Omega\rangle\langle\Omega|q'\rangle$$

$$\Rightarrow \langle\Omega|T\left\{\prod_{k=1}^{n}\hat{q}(t_k)\right\}|\Omega\rangle = \lim_{T\to\infty} \mathcal{N} \frac{e^{2iE_\Omega(1-i\epsilon)T}}{\langle q''|\Omega\rangle\langle\Omega|q'\rangle}$$

$$\times \int[\mathcal{D}q]\, e^{iS} \prod_{k=1}^{n} q(t_i)$$

$$\Rightarrow \quad \frac{\langle \Omega | T\{\prod_{i=1}^{n} \hat{q}(t_i)\} | \Omega\rangle}{\langle \Omega | \Omega \rangle} = \frac{\int [\mathcal{D}q]\, e^{iS} \prod_{i=1}^{n} q(t_i)}{\int [\mathcal{D}q]\, e^{iS}}$$

$i\epsilon$ prescription is needed in order to establish the relation between ~~the~~ path integral definition of correlation funcn and canonical definition.

Effect of $i\epsilon$ prescription on path integral :

~~Def~~ Devide the time interval in 2N segments

$$\Delta = \frac{2T}{2N} = \frac{T}{N}$$

$i\epsilon$ prescription : $T \to T(1-i\epsilon) \Rightarrow \Delta \to \frac{T}{N}(1-i\epsilon) = \Delta_0(1-i\epsilon)$

$$iS = i\sum_{K=-N}^{N}\left[\frac{1}{2}m\frac{(q_{k+1}-q_k)^2}{\Delta_0}(1+i\epsilon) - V(q_k)\,\Delta_0\,(1-i\epsilon)\right]$$

$$= iS - \epsilon\underbrace{\sum_{K=-N}^{N}\left[\frac{1}{2}m\frac{(q_{k+1}-q_k)^2}{\Delta_0} + V(q_k)\,\Delta_0\right]}$$

↳ damping factor.

↳ Suppressed for large $(q_{k+1}-q_k)$

$\Rightarrow$ Kinky paths are suppressed.

Generalization to Multiple q_i

$$H = \sum_{i=1}^{N} \left[\frac{1}{2m}\dot{q}_i^2 + V(q_i)\right] \rightarrow \text{Multipartical q.m.} \qquad S = \int dt \sum_i \left(\frac{1}{2}\dot{\varphi}_i^2 + V(q_i)\right)$$

let us discretise the space

$$\vec{x} = h\,(i_1, i_2, i_3) = h\,\vec{i} \quad ; \quad (i_1, i_2, i_3) = \{-\infty, \infty\}$$

$$\Phi(\vec{x}, t) = \phi_{(i_x, i_y, i_z)}(t) \equiv \phi_{\vec{i}}(t)$$

$$S = \int dt \int d^3x \left[\left(\frac{\partial\varphi}{\partial t}\right)^2 - (\nabla\varphi)^2 - m^2\varphi^2\right]$$

$$= \int dt \quad h^3 \sum_{i_x, i_y, i_z} \left(\partial_0 \phi_{\vec{i}}\right)^2 - \frac{1}{2}\left(\frac{\phi_{i_x+1, i_y, i_z} - \phi_{i_x, i_y, i_z}}{h}\right)^2$$

$$- \cdots - \frac{1}{2}m^2\phi_{\vec{i}}^2 \;\Big]$$

This system is equivalent to a quantum mechanical system of infinite d.o.f.

Now we would like to calculate the correlation func^n.

$$G^n(x_1, \ldots, x_n) = \frac{\langle\Omega| T\{\prod_{a=1}^{n} \hat{\Phi}(x_a)\} |\Omega\rangle}{\langle\Omega|\Omega\rangle} \qquad \hat{\Phi}(x_a) = \hat{\Phi}(\vec{x}_{[a]}, t_a)$$

We again discretise the space $\vec{x}_{[a]} \rightarrow h\,\vec{i}_{[a]}$

$$\hat{\varphi}(\vec{x}_{[a]}, t_{[a]}) = \hat{\varphi}_{\vec{i}_{[a]}}(t_a)$$

$$G^{(n)}(x_1, \cdots, x_n) = \frac{\langle\Omega| T\{\prod_{a=1}^{n} \hat{\varphi}_{\vec{i}_{[a]}}(t_a)\} |\Omega\rangle}{\langle\Omega|\Omega\rangle}$$

Now ~~φ~~ $\phi(\vec{x},t)$'s are dynamical variables. and we need to define a path in $\phi(\vec{x},t)$ space. Like in quantum mechanics we defined a path in q space. In q.m. a path ~~was~~ depends on t, since $q(t)$ ~~was~~ is a func^n of t only (i.e. t was the only parameter to parametrise a path). Thus sum over all path means $\to \int dq_2\, dq_3 \cdots dq_N$ where $q_2, q_3, \cdots$ etc are $q(t_2), q(t_3), \cdots$

Now in field theory, variables are $\phi(\vec{x},t)$ i.e. a variable ~~is~~ depends on 4 parameters $\vec{x}$, and t. Therefore when we specify a path in field space, a path is characterised by four variables. Therefore ~~a path in ϕ space~~ sum over path in ϕ-space can be written as

$$\int \prod_{i_x} \prod_{i_y} \prod_{i_z} \prod_{i_0} d\phi_{\vec{i}, i_0}$$. for example in (1+1) dimension

$$\int d\phi_{1,1}\, d\phi_{12}\, d\phi_{1,3} \cdots d\phi_{21}\, d\phi_{22}\, d\phi_{23} \cdots$$

$$\phi_{12} \to \phi(x=1, t=2) \quad \text{etc.}$$

Therefore, the numerator can be written as

$$\int \prod_{i_x=-\infty}^{\infty} \prod_{i_y=-\infty}^{\infty} \prod_{i_z=-\infty}^{\infty} \prod_{i_0=-\infty}^{\infty} d\phi_{\vec{i}, i_0}\; e^{iS(\phi_{\vec{i}\, i_0})} \prod_{a=1}^{n} \phi_{\vec{i}_{[a]}\, i_{0[a]}}$$

$$\phi_{\vec{i}_{[a]}\, i_{0[a]}} \to \phi(\vec{i}_{[a]}, i_{0[a]})$$

In the continuum limit

$$\int [\mathcal{D}\phi]\, e^{iS[\phi, \partial\phi]}\, \phi(x_1) \cdots \phi(x_n)$$

$$S = h^3 \Delta \sum_{i_0=-\infty}^{\infty} \sum_{\vec{i}=-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{\phi_{\vec{i},i_0+1} - \phi_{\vec{i},i_0}}{\Delta}\right)^2 - \frac{1}{2}\left(\frac{\phi_{\vec{i}+\vec{1},i_0} - \phi_{\vec{i},i_0}}{h}\right)^2 - \frac{1}{2} m^2 \phi_{\vec{i},i_0}^2 - V(\phi_{\vec{i},i_0}) \right]$$

Remember that time axis is tilted by an amount $i\epsilon$. After doing the calculation we take $\epsilon \to 0$ limit.

In the continuum limit

$$\frac{\left\langle T\left\{ \prod_{a=1}^{n} \phi(x_a) \right\} \right\rangle}{\langle \Omega | \Omega \rangle} = \frac{\int [\mathcal{D}\phi]\, e^{iS} \prod_{a=1}^{n} \phi(x_a)}{\int [\mathcal{D}\phi]\, e^{iS}}$$

Functional derivative

$$F[\phi] = \text{functional of } \phi(x)$$

On space-time lattice $F[\phi]$ becomes ordinary func^n of infinite variables.

$$F[\phi(\vec{x},t)] = F[\phi_{\vec{i},i_0}]$$

Define $\dfrac{\delta}{\delta\phi(\vec{x},t)} = \dfrac{1}{\Delta h^3}\dfrac{\partial}{\partial\phi_{\vec{i},i_0}}$ $\qquad \phi_{\vec{i},i_0} \rightarrow$ ordinary variable

$$F[\phi, J] = \int d^4x' \, \phi(x') J(x')$$

$$= \Delta h^3 \sum_{i_0'} \sum_{\vec{i}'} \phi_{\vec{i}', i_0'} J_{\vec{i}', i_0'}$$

$$\frac{\delta F}{\delta\phi(\vec{x},t)} = \frac{1}{\Delta h^3}\cdot\frac{\delta}{\delta\phi_{\vec{i},i_0}}\Delta h^3 \sum_{\vec{i}'}\sum_{i_0'} \phi_{\vec{i}', i_0'} J_{\vec{i}', i_0'}$$

$$= \sum_{\vec{i}', i_0'} \delta_{\vec{i}\,\vec{i}'}\, \delta_{i_0 i_0'}\, J_{\vec{i}', i_0'} = J_{\vec{i}\, i_0}$$

$$= J(x).$$

$$\frac{\delta\phi(x)}{\delta\phi(x')} = \frac{1}{\Delta h^3}\cdot\frac{\partial}{\partial\phi_{\vec{i}', i_0'}}\phi_{\vec{i}, i_0} = \frac{1}{\Delta h^3}\cdot\delta_{\vec{i}\vec{i}', i_0 i_0'} = \delta^4(x-x')$$

Our next goal is to calculate $G^{(n)}(x_1, x_2, \cdots, x_n)$

$$G^n(x_1, \cdots, x_n) = \int [\mathcal{D}\phi]\, e^{iS} \prod_{a=1}^{n} \phi(x_a) \Big/ \int [\mathcal{D}\phi]\, e^{iS}$$

To calculate this we apply a trick

$$Z[J] = \int [\mathcal{D}\phi]\, e^{iS + i\int d^4x\, \phi(x) J(x)}$$

$$-i \frac{\delta}{\delta J(x_1)} Z[J] = -i \int [\mathcal{D}\phi]\, e^{iS[\phi] + i\int d^4x\, \phi(x) J(x)} \times i \int d^4x\, \phi(x) \frac{\delta J(x)}{\delta J(x_1)}$$

$$= \int [\mathcal{D}\phi]\, e^{iS + i\int d^4x\, \phi(x) J(x)}\, \phi(x_1)$$

Illy $$\left(-i \frac{\delta}{\delta J(x_1)}\right) \cdots \left(-i \frac{\delta}{\delta J(x_n)}\right) Z[J] = \int [\mathcal{D}\phi]\, e^{iS + i\int d^4x\, \phi(x) J(x)} \times \phi(x_1)\phi(x_2)\cdots\phi(x_n)$$

Thus $$\left[\prod_{a=1}^{n} \left(-i \frac{\delta}{\delta J(x_a)}\right) Z[J]\right]_{J=0} = \int [\mathcal{D}\phi]\, e^{iS}\, \phi(x_1)\cdots\phi(x_n)$$

And $$Z[0] = \int \mathcal{D}[\phi]\, e^{iS}$$

$$\therefore \boxed{G^n(x_1, \cdots, x_n) = \frac{\prod_{a=1}^{n} \left(-i \frac{\delta}{\delta J(x_a)} Z[J]\right)\Big|_{J=0}}{Z[0]}}$$

Example:

1. Free scalar theory.

$$S = \frac{1}{2}\int d^4x \left[\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right]$$

$$= -\frac{1}{2}\int d^4x\; \phi(x)\,(\Box + m^2)\,\phi(x)$$

$$S + \int d^4x\; \phi(x)\,J(x)$$

$$= \frac{1}{2}\int d^4x \left[-\phi(x)\,(\Box+m^2)\,\phi(x) + 2J(x)\,\phi(x)\right]$$

Now $\quad (\Box+m^2)\,\Delta_F(x-x') = -i\,\delta^4(x-x')$

Claim: $\quad S + \int J(x)\,\phi(x)\,d^4x$

$$= -\frac{1}{2}\int d^4x \left\{\phi(x) - i\int d^4x'\,\Delta_F(x-x')\,J(x')\right\} (\Box+m^2)\left\{\phi(x) - i\int d^4x''\,\Delta_F(x-x'')\,J(x'')\right\}$$

$$+ \frac{i}{2}\int d^4x'\,d^4x\; J(x)\,\Delta_F(x-x')\,J(x')$$

Ex 20: Prove the above eqn.

Define: $\quad \chi(x) = \phi(x) - i\int d^4x'\,\Delta_F(x-x')\,J(x')$

$$\Rightarrow\quad S + \int d^4x\,\phi(x)\,J(x) = -\frac{1}{2}\int d^4x\;\chi(x)\,(\Box+m^2)\,\chi(x) + \frac{i}{2}\int d^4x'\,d^4x\; J(x)\,\Delta_F(x-x')\,J(x')$$

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$$[\mathcal{D}\phi] = d\varphi_1\, d\varphi_2 \cdots d\varphi_N$$

$$\phi(x) = \chi(x) + f(x)$$

$$\phi_1 = \chi_1 + f_1$$

$$d\varphi_1 = d\chi_1$$

$$d\varphi_2 = d\chi_2$$

$$d\varphi_1 d\varphi_2 \cdots d\varphi_N = d\chi_1\, d\chi_2 \cdots d\chi_N = [\mathcal{D}\chi]$$

$$[\mathcal{D}\phi] = [\mathcal{D}\chi]$$

$$Z[J] = \int [\mathcal{D}\phi]\, e^{i\left[S + \int d^4x\, \phi(x) J(x)\right]}$$

$$= \int [\mathcal{D}\chi]\, e^{-\frac{i}{2}\int d^4x\, \chi(x)(\Box + m^2)\chi(x)}$$

$$\times\, e^{-\frac{1}{2}\int d^4x\, d^4x'\, J(x)\, \Delta_F(x-x')\, J(x')}$$

$$= Z[0] \exp\left[-\frac{1}{2}\int d^4x\, d^4x'\, J(x)\, \Delta_F(x-x')\, J(x')\right]$$

$$G^2(x_1, x_2) = \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J(x_1)}\right)\left(-i \frac{\delta}{\delta J(x_2)}\right) Z[J]\Bigg|_{J=0}$$

$$= \frac{1}{Z[0]} (-i)^2 \frac{\delta}{\delta J(x_1)} \Bigg[e^{-\frac{1}{2}\int d^4x d^4x'\, J(x)\, \Delta_F(x-x')\, J(x')}$$

$$\Big\{ -\frac{1}{2}\int d^4x\, J(x)\, \Delta_F(x-x')\, \delta^4(x'-x_2)\, d^4x'$$

$$-\frac{1}{2}\int d^4x\, \delta^4(x-x_2)\, \Delta_F(x-x')\, J(x') \Big\} \Bigg] Z[0] \Bigg|_{J=0}$$

$$= -1 \frac{\delta}{\delta J(x_1)} \Bigg[e^{-\frac{1}{2}\int d^4x d^4x'\, J(x)\, \Delta_F(x-x')\, J(x')} \Big\{ -\int d^4x\, J(x)\, \Delta_F(x-x_2) \Big\} \Bigg]\Bigg|_{J=0}$$

$$= -\frac{\delta}{\delta J(x_1)} \left[\left\{ -\int d^4x\, J(x)\, \Delta_F(x-x_2)\right\} \frac{Z[J]}{Z[0]} \right]$$

$$= -1 \Bigg[e^{-\frac{1}{2}\int d^4x\, d^4x'\, J(x)\, \Delta_F(x-x')\, J(x')} \left\{ -\int d^4x\, \delta^4(x-x_1)\, \Delta_F(x-x_2) \right\}$$

$$+ e^{-\frac{1}{2}\int d^4x\, d^4x'\, J(x)\, \Delta_F(x-x')\, J(x')} \left\{ -\frac{1}{2}\int d^4x\, J(x)\, \Delta_F(x-x_1) \right\} \left\{ -\int d^4x\, J(x)\, \Delta_F(x-x_2) \right\} \Bigg]_{J=0}$$

$$= -\left[-\Delta_F(x_1-x_2) + 0 \right]$$

$$= \Delta_F(x_1-x_2)$$

Ex: 21 Calculate 4 point func^n in free theory. Show that

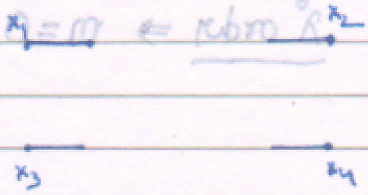
$$G^4(x_1, x_2, x_3, x_4) = \Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\,\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3)$$

Ex 22 Show that $G^3(x_1, x_2, x_3) = \langle 0|\, T\{\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\}\, |0\rangle = 0$

Introduce a diagrammatic technique (Feynman Diagram)

$$\langle \varphi(x_1)\cdots\varphi(x_n)\rangle = \int [\mathcal{D}\varphi]\, e^{iS}\, \varphi(x_1)\cdots\varphi(x_n) \Big/ \int \mathcal{D}\varphi\, e^{iS}$$

$$= \frac{\left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right) Z_f[J]}{Z_f[0]}\Bigg|_{J=0}$$



For each $\left(-i\frac{\delta}{\delta J(x_i)}\right) \longrightarrow$ x_i ___ $\varphi(x_i)$

- Then join all these lines w/ each other in all possible ways.
- A line joins 2 points → represented by $\Delta_F(x_i - x_j)$