(12)

Propagation of signal — ~~Causality~~.

Consider states in position base

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle$$

$$\langle \vec{q}|\underbrace{\phi(x)|0\rangle}_{} \approx e^{-i\vec{q}\cdot\vec{x}} \qquad \Big| \text{Recall } \langle p|x\rangle = e^{-i\vec{p}\cdot\vec{x}}$$

↳ $\phi(x)$ acting on vacuum creats a particle at x.

Therefore consider the probability for a particle propagating from $x = (\vec{x}, t)$ to $y = (\vec{y}, t')$

$$\langle x|y\rangle = \langle 0|\phi(x)\,\phi(y)|0\rangle \equiv D_1(x,y)$$

~~However in QFT we define something more fundamental — Correlation func^n.~~

$$\cancel{\Delta(x,y) = \langle 0|[\phi(x),\phi(y)]|0\rangle}$$

$$\cancel{= D_1(x,y) - D_1(y,x)}$$

$$D_1(x,y) = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \cdot \frac{1}{\sqrt{2\omega_{\vec{p}}}} \frac{1}{\sqrt{2\omega_{\vec{q}}}} \langle 0|a_{\vec{p}}\, e^{-ip\cdot x}\, e^{iqy}\, a^{+}_{q}|0\rangle$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \cdot \frac{1}{2\sqrt{\omega_{\vec{p}}\,\omega_{\vec{q}}}} (2\pi)^3 \delta^3(\vec{p}-\vec{q})\, e^{-ip\cdot x + iqy}$$

$$= \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2\omega_{\vec{p}}} e^{-ip(x-y)}$$

$$\equiv D_1(x-y)$$

In QFT we define a more fundamental quantity

~~correlation function~~

~~Two point~~ $\Delta(x-y) = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$

$$= D_1(x-y) - D_1(y-x)$$

$D_1(x-y) = \langle 0 | \phi(x)\, \phi(y) | 0 \rangle \longrightarrow$ Lorentz invariant

Ground states are Lorentz invariant also $\phi(x)$, $\phi(y)$ are Lorentz invariant.
Since $D_1(x-y)$ is amplitude for a particle to propogate from y to x and this amplitude should not depend on any ~~parti~~ Lorentz frame.

Now let us define a quantity

$$\Delta_F(x-y) = \lim_{\epsilon \to 0} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

Now $p^2 - m^2 + i\epsilon = p_0^2 - (\vec{p}^2 + m^2) + i\epsilon$

$$= p_0^2 - (\omega_p^2 - i\epsilon)$$

The integrand has a pole at $p_0^2 = \omega_p^2 - i\epsilon = \omega_p^2 \left(1 - \frac{i\epsilon}{\omega_p^2}\right)$

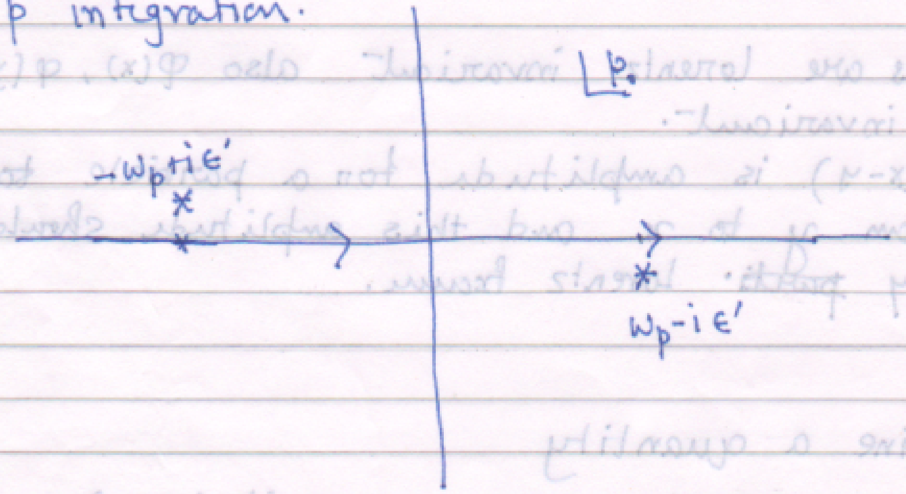
$$p_0 = \pm \omega_p \left(1 - \frac{i\epsilon}{2\omega_p^2}\right)$$

$$= \pm (\omega_p - i\epsilon')$$

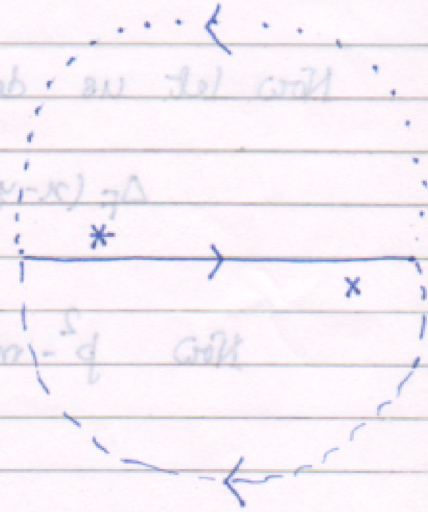
$$\Delta_F(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{dp_0}{(2\pi)} \frac{i}{[p_0-(\omega_p-i\epsilon')][p_0+(\omega_p-i\epsilon')]} e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left\{ \int \frac{dp_0}{2\pi} \frac{i\, e^{-ip_0(x^0-y^0)}}{[p_0-(\omega_p-i\epsilon')][p_0+(\omega_p-i\epsilon')]} \right\} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

Lets do p^0 integration.



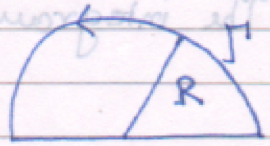
How do we close the contour ?



Two possibilities → upper half plane
→ lower half plane

Jordon's Lemma.

$$\lim_{R\to\infty} \int_\Gamma e^{iaz} f(z)\, dz \longrightarrow 0 \quad \text{if } a>0$$



and if $f(z)$ goes to zero faster than $\frac{1}{|z|}$ for $\arg(z) \in [0,\pi]$ as $z\to\infty$.

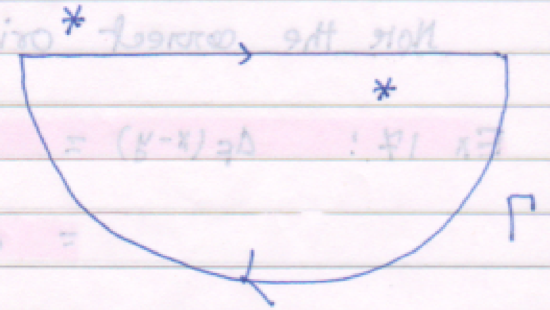
If $a<0$ the theorem also holds, in that case we need to ~~close~~ consider Γ in LHP.

$$\int \frac{dp_0}{2\pi} \frac{i e^{-ip_0(x^0-y^0)}}{\left[p_0-(\omega_p-i\epsilon')\right]\left[p_0+(\omega_p-i\epsilon')\right]}$$

Let $x^0 > y^0$.

Here $a = -(x^0-y^0) < 0$

Hence, $p^0 = \omega_p - i\epsilon'$ will contribute.



Residue at $p^0 = \omega_p - i\epsilon'$

$$\frac{1}{2\pi} \frac{i\, e^{-i(\omega_p - i\epsilon')(x^0-y^0)}}{2\omega_p - 2i\epsilon'} = \frac{i}{2\pi} \frac{e^{-i\omega_p(x^0-y^0)} \cdot e^{-\epsilon'(x^0-y^0)}}{2(\omega_p - i\epsilon')}$$

$$\therefore \lim_{\epsilon'\to 0} \int \frac{dp^0}{2\pi}\, i \frac{e^{-ip_0(x^0-y^0)}}{[\;\;][\;\;]} = (-)2\pi i \frac{i}{2\pi} \frac{e^{-i\omega_p(x^0-y^0)}}{2\omega_p}$$

Because here the conture is clockwise $= + \dfrac{e^{-i\omega_p(x^0-y^0)}}{2\omega_p}$

$$\therefore \Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-i\omega_p(x^0-y^0) + i\vec{p}\cdot(\vec{x}-\vec{y})} \quad \text{for } x^0 > y^0$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip\cdot(x-y)} \quad \text{for } x^0 > y^0$$

$$= D_1(x-y) \quad \text{for } x^0 > y^0. = \langle 0|\phi(x)\phi(y)|0\rangle \quad \text{for } x^0 > y^0$$

$\Delta_F(x-y) \rightarrow$ Probability of a particle going from y to x when $y^0 < x^0$.

n-point correlation func^n:

$$G_n(x_1, x_2, \cdots, x_n) = \langle 0 | T\{\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\} | 0 \rangle$$

$$G_2(x_1, x_2) = \Delta_F(x_1 - x_2) \longrightarrow \text{Propagator.}$$

Q. What is the importance of n-point correlators. ?

n-point correlation functions contain information about n-particle scattering process.

In practice we would like to apply QFT to calculate scattering crossection of different process. It is possible to show that scattering amplitudes are related (proportional) to n-point Green's func^n.

Goal: Our goal is to calculate n-point func^{n}s in free field theory.

Lets write $\varphi(x) = \varphi^+(x) + \varphi^-(x)$

$$\begin{cases} \varphi^+(x) = \int \frac{d^3p}{(2\pi)^3}\, a_p\, e^{-ip\cdot x} \frac{1}{\sqrt{2\omega_p}} \\ \varphi^-(x) = \int \frac{d^3p}{(2\pi)^3}\, a_p^\dagger\, e^{ip\cdot x} \frac{1}{\sqrt{2\omega_p}} \end{cases}$$

$$\varphi^+(x)|0\rangle = 0$$

$$\langle 0|\varphi^-(x) = 0$$

Now consider $T\{\varphi(x)\varphi(y)\}$ and consider a particular time ordering say $x^0 > y^0$:

$$\begin{aligned}
T\{\varphi(x)\varphi(y)\} = \varphi(x)\varphi(y) &= (\varphi^+(x) + \varphi^-(x))(\varphi^+(y) + \varphi^-(y)) \\
&= \varphi^+(x)\varphi^+(y) + \varphi^+(x)\varphi^-(y) \\
&\quad + \varphi^-(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y) \\
&= \varphi^+(x)\varphi^+(y) + \varphi^-(y)\varphi^+(x) \\
&\quad + \varphi^-(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y) \\
&\quad + \varphi^+(x)\varphi^-(y) - \varphi^-(y)\varphi^+(x) \\
&= \{\varphi^+(x)\varphi^+(y) + \varphi^-(y)\varphi^+(x) + \varphi^-(x)\varphi^+(y) + \varphi^-(x)\varphi^-(y)\} \\
&\quad + [\varphi^+(x), \varphi^-(y)]
\end{aligned}$$

In all the terms in { } bracket, φ^+ is sitting right to φ^-.
→ They are called normal ordered.

$$\{\varphi^+(x)\varphi^+(y) + \varphi^-(x)\varphi^+(y) + \varphi^-(y)\varphi^+(x) + \varphi^-(x)\varphi^-(y)\}$$

$$= N(\varphi(x)\varphi(y)) \equiv :(\varphi(x)\varphi(y)):$$

Illy for $y^0 > x^0$

$$T\{\varphi(x)\varphi(y)\} = \varphi(y)\varphi(x)$$

$$= \varphi^+(y)\varphi^+(x) + \varphi^+(y)\varphi^-(x) + \varphi^-(y)\varphi^+(x) + \varphi^-(y)\varphi^-(x)$$

$$= N\{\varphi(x)\varphi(y)\} + [\varphi^+(y), \varphi^-(x)]$$

$$\therefore T\{\varphi(x)\varphi(y)\} = N\{\varphi(x)\varphi(y)\} + \Theta(x^0 - y^0)[\varphi^+(x), \varphi^-(y)] + \Theta(y^0 - x^0)[\varphi^+(y), \varphi^-(x)]$$

$$= N\{\varphi(x)\varphi(y)\} + \varphi(x)\varphi(y)$$

Where $\varphi(x)\varphi(y) = \Theta(x^0 - y^0)[\varphi^+(x), \varphi^-(y)] + \Theta(y^0 - x^0)[\varphi^+(y), \varphi^-(x)]$

In general: Wick's theorem

$$T\{\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\}$$

$$= N\{\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n) + \text{All possible connections}\}.$$

For Proof – check of Peskin

Example: $T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}$

$$= N\Big\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)$$

$$+ \wick{\c1\varphi(x_1)\c1\varphi(x_2)}\ \varphi(x_3)\varphi(x_4)$$

$$+ \wick{\c1\varphi(x_1)\c1\varphi(x_3)}\ \varphi(x_2)\varphi(x_4)$$

$$+ \wick{\c1\varphi(x_1)\c1\varphi(x_4)}\ \varphi(x_2)\varphi(x_3)$$

$$+ \cdots$$

$$\cdots$$

$$+ \wick{\c1\varphi(x_1)\c1\varphi(x_2)\ \c2\varphi(x_3)\c2\varphi(x_4)}$$

$$+ \wick{\c1\varphi(x_1)\c2\varphi(x_2)\c1\varphi(x_3)\c2\varphi(x_4)}$$

$$+ \wick{\c1\varphi(x_1)\c2\varphi(x_2)\c2\varphi(x_3)\c1\varphi(x_4)}\Big\}$$

$$= N\{1234\} + \wick{\c1 1\c1 2}\,N(34) + \wick{\c1 1\c1 3}\,N(24) + \wick{\c1 1\c1 4}\,N(23)$$

$$+ \wick{\c1 2\c1 3}\,N(14) + \wick{\c1 2\c1 4}\,N(13) + \wick{\c1 3\c1 4}\,N(12)$$

$$+ \wick{\c1 1\c1 2}\ \wick{\c1 3\c1 4} + \wick{\c1 1\c1 3}\ \wick{\c1 2\c1 4} + \wick{\c1 1\c1 4}\ \wick{\c1 2\c1 3}$$

Let us consider

$$\overbrace{\varphi(x)\varphi(y)} = \langle 0|\, \overbrace{\varphi(x_0)\,\varphi(y_0)}\,|0\rangle$$

$$= \theta(x^0-y^0)\,\langle 0|\,[\varphi^+(x), \varphi^-(y)]\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,[\varphi^+(y), \varphi^-(x)]\,|0\rangle$$

$$= \theta(x^0-y^0)\,\langle 0|\,\varphi^+(x)\,\varphi^-(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\varphi^+(y)\,\varphi^-(x)\,|0\rangle$$

$$= \theta(x^0-y^0)\,\langle 0|\,\varphi(x)\,\varphi(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\varphi(y)\,\varphi(x)\,|0\rangle$$

$$= \Delta_F(x-y)$$

$$\therefore\quad \langle 0|\,T\{\varphi(x)\varphi(y)\}\,|0\rangle$$

$$= \langle 0|\,N\{\varphi(x)\varphi(y)\}\,|0\rangle + \Delta_F(x-y)$$

$$= \Delta_F(x-y)$$

$$\langle 0|\,T\{\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4)\}\,|0\rangle$$

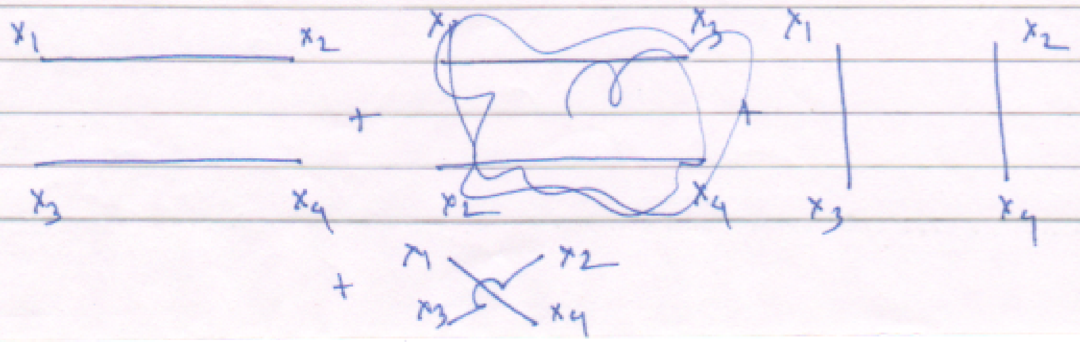
$$= \langle 0|\,N\{\varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4)\}\,|0\rangle$$

$$\cancel{+ \langle 0|\,N\{\overbrace{\varphi(x_1)\,\varphi(x_2)}\,\varphi(x_3)\,\varphi(x_4)\}\,|0\rangle + \cdots}$$

$$+ \overbrace{\varphi(x_1)\,\varphi(x_2)}\,\langle 0|\,\varphi(x_3)\,\varphi(x_4)\,|0\rangle + \cdots$$

$$+ \Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4) + \Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3)$$

$$+ \Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3)$$



$$\langle 0 | T\{\phi(x_1)\phi(x_2)\phi(x_3)\} | 0\rangle$$

Ex 19: $$\langle 0 | T\{\phi(x_1)\phi(x_2) :\phi^2(x_3):\} | 0\rangle$$

Calculate (applying Wick's theorem)

$$\langle 0 | T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,\phi^4(x)\} | 0\rangle$$

$$\langle 0 | T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,:\phi^4(x):\} | 0\rangle.$$

→ Draw the Feynman diagrams.