Path integral for Dirac fields

$$\left\langle \prod_{i=1}^{m} \psi_{\alpha_i}(x_i) \prod_{j=1}^{n} \bar{\psi}_{\beta_j}(y_j) \right\rangle$$

$$= \frac{\int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\, e^{iS} \prod_{i=1}^{m} \psi_{\alpha_i}(x_i) \prod_{j=1}^{n} \bar{\psi}_{\beta_j}(y_j)}{\int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\, e^{iS}}$$

$$S = \int d^4x\, \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \qquad \text{(free action)}.$$

In momentum space

$$\left\langle \prod_{i=1}^{m} \tilde{\psi}_{\alpha_i}(k_i) \prod_{j=1}^{n} \tilde{\bar{\psi}}_{\beta_j}(p_j) \right\rangle$$

$$= \frac{\int [\mathcal{D}\tilde{\psi}][\mathcal{D}\tilde{\bar{\psi}}]\, e^{iS[\tilde{\psi}, \tilde{\bar{\psi}}]} \prod_{i=1}^{m} \tilde{\psi}_{\alpha_i}(k_i) \prod_{j=1}^{n} \tilde{\bar{\psi}}_{\beta_j}(p_j)}{\int [\mathcal{D}\tilde{\psi}][\mathcal{D}\tilde{\bar{\psi}}]\, e^{iS}}$$

Define: $$\tilde{Z}[\tilde{J}, \tilde{\bar{J}}] = \int [\mathcal{D}\tilde{\psi}][\mathcal{D}\tilde{\bar{\psi}}] \exp\left[iS + i\int \frac{d^4k}{(2\pi)^4} \tilde{\bar{J}}_\alpha(-k)\, \tilde{\psi}_\alpha(k) + i\int \frac{d^4p}{(2\pi)^4} \tilde{\bar{\psi}}_\beta(p)\, \tilde{J}_\beta(-p) \right]$$

Then $$\left\langle \prod_{i=1}^{m} \tilde{\psi}_{\alpha_i}(k_i) \prod_{j=1}^{n} \tilde{\bar{\psi}}_{\beta_j}(p_j) \right\rangle$$

$$= \frac{1}{\tilde{Z}[0,0]} \left\{ -i(2\pi)^4 \frac{\delta}{\delta \tilde{\bar{J}}_{\alpha_1}(-k_1)} \right\} \cdots \left\{ -i(2\pi)^4 \frac{\delta}{\delta \tilde{\bar{J}}_{\alpha_m}(-k_m)} \right\} \left\{ i(2\pi)^4 \frac{\delta}{\delta \tilde{J}_{\beta_1}(-p_1)} \right\} \cdots \left\{ i(2\pi)^4 \frac{\delta}{\delta \tilde{J}_{\beta_n}(-p_n)} \right\} \tilde{Z}[\tilde{J}, \tilde{\bar{J}}]$$

Free theory

$$S_f = \int d^4x\ \bar{\psi}(x)\left(i\partial\!\!\!/ - m\right)\psi(x)$$

$$\partial_\mu e^{-ik\cdot x} = -ik_\mu e^{-ik\cdot x}$$

$$\psi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\,\tilde{\psi}(-k)$$

$$\bar{\psi}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\,\tilde{\bar{\psi}}(k)$$

$$S_f = \int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, e^{-i(k+q)x}\ \tilde{\bar{\psi}}_\alpha(q)\left(k\!\!\!/ - m\right)_{\alpha\beta}\tilde{\psi}_\beta(-k)$$

$$= \int \frac{d^4k}{(2\pi)^4}\ \tilde{\bar{\psi}}_\alpha(k)\left(k\!\!\!/ - m\right)_{\alpha\beta}\tilde{\psi}_\beta(-k) \longrightarrow \int \frac{d^4k}{(2\pi)^4}\ \tilde{\bar{\psi}}_\alpha(k)\left(k\!\!\!/ - m + i\epsilon\right)\tilde{\psi}_\beta$$

$$S_f + \int \frac{d^4k}{(2\pi)^4}\left[\tilde{\bar{J}}_\alpha(-k)\,\tilde{\psi}_\alpha(k) + \tilde{\bar{\psi}}_\beta(k)\,\tilde{J}_\beta(-k)\right]$$

$$= \int \frac{d^4k}{(2\pi)^4}\left[\tilde{\bar{\psi}}_\alpha(k)\left(k\!\!\!/ - m\right)_{\alpha\beta}\tilde{\psi}_\beta(-k) + \tilde{\bar{J}}_\alpha(-k)\,\tilde{\psi}_\alpha(k) + \tilde{\bar{\psi}}_\beta(k)\,\tilde{J}_\beta(-k)\right]$$

$$= \int \frac{d^4k}{(2\pi)^4}\Big[\left\{\tilde{\bar{\psi}}_\alpha(k) + (k\!\!\!/ - m)^{-1}_{\alpha\gamma}\tilde{\bar{J}}_\gamma(k)\right\}\left(k\!\!\!/ - m\right)_{\alpha\beta}\left\{\tilde{\psi}_\beta(-k) + (k\!\!\!/ - m)^{-1}_{\beta\delta}\tilde{J}_\delta(-k)\right\}$$
$$- \tilde{\bar{J}}(k)\left(k\!\!\!/ - m\right)^{-1}\tilde{J}(-k)\Big]$$

$$= \int \frac{d^4k}{(2\pi)^4}\left[\tilde{\bar{\chi}}(k)\left(k\!\!\!/ - m\right)\tilde{\chi}(-k) - \tilde{\bar{J}}(k)\left(k\!\!\!/ - m\right)^{-1}\tilde{J}(-k)\right]$$

$[\mathcal{D}\tilde{\psi}] = [\mathcal{D}\tilde{\chi}]$ and $[\mathcal{D}\tilde{\bar{\psi}}] = [\mathcal{D}\tilde{\bar{\chi}}]$

$$\tilde{Z}_f[\tilde{\eta}, \tilde{\bar{J}}] = \int [\mathcal{D}\tilde{\chi}][\mathcal{D}\tilde{\bar{\psi}}]\, e^{iS_f[\tilde{\chi}, \tilde{\bar{\chi}}]}\, e^{-i\int \frac{d^4k}{(2\pi)^4} \tilde{\bar{J}}(k)(\not{k}-m)^{-1}\tilde{J}(-k)}$$

$$= \tilde{Z}_f[0,0]\, \exp\left[-\int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\, \tilde{\bar{J}}_{\alpha'}(-k')\, (\tilde{S}_F)_{\alpha\beta}\, \tilde{J}_{\beta}(-k)\right]$$

$$(\tilde{S}_F)_{\alpha\beta}(k,k') = \left(\frac{i}{\not{k}-m+i\epsilon}\right)_{\alpha\beta} (2\pi)^4 \delta^4(k+k')$$

<u>Two point func^n</u>

$$\langle \tilde{\psi}_\alpha(k_1)\, \tilde{\bar{\psi}}_\beta(k_2)\rangle$$

$$= \frac{1}{\tilde{Z}_f[0,0]} \left\{-i(2\pi)^4 \frac{\delta}{\delta \tilde{\bar{J}}_\alpha(-k_1)}\right\} \left\{ i(2\pi)^4 \frac{\delta}{\delta \tilde{J}_\beta(-k_2)}\right\} \tilde{Z}_f[\tilde{J}, \tilde{\bar{J}}] \Bigg|_{\substack{\tilde{J}=0 \\ \tilde{\bar{J}}=0}}$$

<u>Ex 83</u> $= (\tilde{S}_F(k_1,k_2))_{\alpha\beta}$

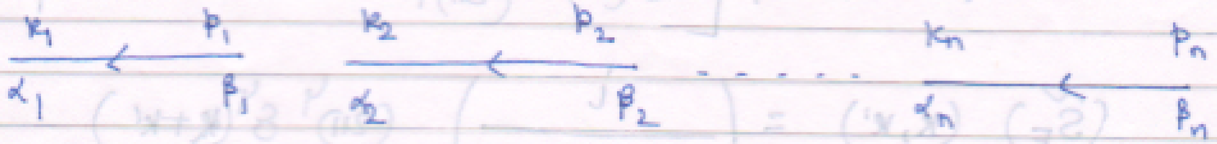
$$= \left(\frac{i}{\not{k}-m+i\epsilon}\right)_{\alpha\beta} (2\pi)^4 \delta^4(k_1+k_2) \qquad \alpha \longleftarrow \beta$$

$$\langle \psi\psi\rangle = 0, \quad \langle \bar{\psi}\bar{\psi}\rangle = 0$$

n-point correlation func^n

$$\langle \tilde{\psi}_{\alpha_1}(k_1)\, \tilde{\bar{\psi}}_{\beta_1}(p_1)\, \tilde{\psi}_{\alpha_2}(k_2)\, \tilde{\bar{\psi}}_{\beta_2}(p_2) \cdots \tilde{\psi}_{\alpha_n}(k_n)\, \tilde{\bar{\psi}}_{\beta_n}(p_n) \rangle$$

=



$$+ (-1)^{N_S} \text{ permutations.}$$

How to calculate N_S?

(i) Count the no. of intersection

(ii) Count how many pairs are in order $\bar{\psi}\psi$ instead of $\psi\bar{\psi}$ = n

$$N_S = m + n$$

$$\langle \tilde{\psi}_{\alpha_1}(k_1)\, \tilde{\psi}_{\alpha_2}(k_2)\, \tilde{\bar{\psi}}_{\beta_1}(p_1)\, \tilde{\bar{\psi}}_{\beta_2}(p_2) \rangle$$

$$= -\left(\tilde{S}_F\right)_{\alpha_1\beta_1}(k_1, p_1)\left(\tilde{S}_F\right)_{\alpha_2\beta_2}(k_2, p_2) + \left(\tilde{S}_{F\alpha_1\beta_2}(k_1, p_2)\, \tilde{S}_{F\alpha_2\beta_1}(k_2, p_1)\right)$$

$$\langle \tilde{\psi}_{\alpha_1}(k_1)\, \tilde{\bar{\psi}}_{\beta_1}(p_1)\, \tilde{\psi}_{\alpha_2}(k_2)\, \tilde{\bar{\psi}}_{\beta_2}(p_2) \rangle$$

$$= \tilde{S}_{F\alpha_1\beta_1}(k_1, p_1)\, \tilde{S}_{F\alpha_2\beta_2}(k_2, p_2) + (-1)^1\, \tilde{S}_{F\alpha_1\beta_2}(k_1, p_2)\, \tilde{S}_{F\alpha_2\beta_1}(k_2, p_1)$$

Ex : 34 $\quad \langle \tilde{\psi}_{\alpha_1}(k_1)\, \tilde{\psi}_{\alpha_2}(k_2)\, \tilde{\bar{\psi}}_{\beta_1}(p_1)\, \tilde{\bar{\psi}}_{\beta_2}(p_2) \rangle$

Lagrangian / action for free fermion

$$S_f = \int d^4x\, \mathcal{L} \qquad \mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x)$$

The action is invariant under $\psi'(x) = e^{ie\lambda}\psi(x)$
where e is a constant and the transformation parameter λ is also independent of space and time.

$$\psi'(x) = e^{ie\lambda}\psi(x) \Rightarrow \psi'^{+}(x) = e^{-ie\lambda}\psi^{+}(x) \Rightarrow \bar{\psi}'(x) = e^{-ie\lambda}\bar{\psi}(x)$$

$$\mathcal{L}' = \bar{\psi}'(x)(i\not{\partial} - m)\psi'(x) = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) = \mathcal{L}$$

Lagrangian is invariant under this transformation.

The transformation is called $\longrightarrow$ Global phase transformation

Q: Can we make the symmetry transformation a local symmetry trasfⁿ?

i.e. $\lambda \longrightarrow \lambda(x)$.

$$\psi'(x) = e^{ie\lambda(x)}\psi(x) \quad \text{and} \quad \bar{\psi}'(x) = e^{-ie\lambda(x)}\bar{\psi}(x)$$

$$\mathcal{L}' = \bar{\psi}'(x)(i\not{\partial} - m)\psi'(x) = e^{-ie\lambda(x)}\bar{\psi}(x)(i\not{\partial} - m)\left[e^{ie\lambda(x)}\psi(x)\right]$$

$$= e^{-ie\lambda(x)}\bar{\psi}(x)\left[e^{ie\lambda(x)}(i\not{\partial} - m)\psi + \left(i\gamma^\mu(ie\partial_\mu\lambda)\right)e^{ie\lambda(x)}\psi(x)\right]$$

$$= \mathcal{L} - e\,\gamma^\mu\partial_\mu\lambda\,\bar{\psi}(x)\psi(x)$$

So Lagrangian is not invariant under local phase transformation.

Let us add an additional term in the Lagrangian/action such that the total action is invariant.

$$S_{tot} = S_f + S_{add}$$

$$= \int d^4x\, \bar{\psi}(x)(i\not{\partial} - m)\psi(x) + \alpha \int d^4x\, \bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x)$$

Our claim is that the new field $A_\mu(x)$ will transform in such a way so that the total action is invariant under ~~e.g.~~ local gauge trasfm.

$$S'_{tot} = \int d^4x\, \bar{\psi}'(x)(i\not{\partial} - m)\psi'(x) + \alpha \int d^4x\, \bar{\psi}'(x)\gamma^\mu A'_\mu(x)\psi'(x)$$

$$= \int d^4x\, \bar{\psi}(x)(i\not{\partial} - m)\psi(x) - e\int d^4x\, \gamma^\mu \partial_\mu \lambda\, \bar{\psi}(x)\psi(x)$$

$$+ \alpha \int d^4x\, \bar{\psi}(x)\gamma^\mu A'_\mu(x)\psi(x)$$

$$= \int d^4x\, \bar{\psi}(x)(i\not{\partial} - m)\psi(x) + \alpha\int d^4x\, \bar{\psi}(x)\underbrace{\left[\gamma^\mu A'_\mu(x) - \frac{e}{\alpha}\gamma^\mu \partial_\mu \lambda\right]}_{\gamma^\mu A_\mu(x)}\psi(x)$$

$$= \int d^4x\, \bar{\psi}(x)(i\not{\partial} - m)\psi(x) + \alpha \int d^4x\, \bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x)$$

$$\boxed{A'_\mu(x) = A_\mu(x) + \frac{e}{\alpha}\partial_\mu \lambda}$$

If we choose $\alpha = e$

The $S_{tot} = \int d^4x \left[\bar{\psi}(x)(i\not{\partial} - m)\psi(x) + e\,\bar{\psi}(x)\gamma^\mu A_\mu \psi(x)\right]$ is invariant

under $\psi'(x) = e^{ie\lambda(x)}\psi(x)$, $\bar{\psi}'(x) = e^{-ie\lambda(x)}\bar{\psi}(x)$

and $\boxed{A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x)}$

$A_\mu(x)$ is called gauge field.

Q. What is the physical meaning of $A_\mu(x)$.

To understand that we need to know the dynamics of this field $A_\mu(x)$.
And dynamics is given by the eqn of motion.

Thus we need to write down a kinetic energy term for the gauge field.

The kinetic energy part shout contain two derivatives and it has to be gauge invariant also.

Let us define $\quad F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu (A_\mu(x))$

$$F'_{\mu\nu}(x) = \partial_\mu A'_\nu(x) - \partial_\nu A'_\mu(x)$$

$$= \partial_\mu A_\nu(x) + \partial_\mu\partial_\nu\lambda(x) - \partial_\nu A_\mu(x) - \partial_\nu\partial_\mu\lambda(x)$$

$$= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) = F_{\mu\nu}(x)$$

Kinetic term is $\quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \qquad F^{\mu\nu}(x) = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta}(x)$

$$S_g = -\frac{1}{4}\int d^4x \; F_{\mu\nu}(x) F^{\mu\nu}(x)$$

Equation of motion $\quad \delta S_g = 0 \quad$ (for pure gauge theory)

Ex : 35 Show that eom is $\boxed{\partial_\mu F^{\mu\nu}(x) = 0}$

$$\partial_\mu F^{\mu\nu}(0) = 0$$

$\nu = 0$

$$\partial_\mu F^{\mu 0} = 0$$

$$\partial_i F^{i0} = 0$$

$$\Rightarrow \quad \partial_i \eta^{ij}\eta^{00} F_{j0} = 0$$

$$\Rightarrow \quad \partial_j F_{j0} = 0$$

$$\Rightarrow \quad \partial_j\left(\partial_j A_0 - \partial_0 A_j\right) = 0$$

$$\cancel{\Rightarrow \nabla^2 A_0 - \partial_t(\vec{\nabla}\cdot\vec{A}) = 0}$$

$\nu = i$

$$\partial_\mu F^{\mu i} = 0$$

$$\partial_0 F^{0i} + \partial_j F^{ji} = 0$$

$$\partial_0 F_{0i} + \partial_j F_{ji} = 0$$

$$-\partial_0 F_{0i} + \partial_j\left(\partial_j A_i - \partial_i A_j\right) = 0$$

Let us identify $A_0 = \phi$, $\{A^i_0\} = \vec{A} = \{A_x, A_y, A_z\} \Rightarrow A_i = -A_i$

Then $F_{j0} = \partial_j A_0 - \partial_0 A_j = \nabla_j \phi - \frac{\partial}{\partial t} A_j = -E_j$ or $\boxed{\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}}$

Thus the first eqn. is $-\partial_j E_j = 0$ or $\boxed{\vec{\nabla}\cdot\vec{E} = 0}$

Also $(\partial_j A_i - \partial_i A_j)\,\epsilon^{ijk} = \cancel{B_k} - (\vec{\nabla}\times\vec{A})_k = -B_k$

$$\Rightarrow \quad \epsilon^{ijk}\left(\partial_i A_j - \partial_j A_i\right) = B_k$$

$$\Rightarrow \quad \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k$$

Then from the second eqn.

$$\partial_0 E_i - \partial_j \epsilon_{ijk} B_k = 0$$

$$\frac{\partial E_i}{\partial t} - (\vec{\nabla}\times\vec{B})_i = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \vec{E}}{\partial t} = \vec{\nabla}\times\vec{B}}$$

Correct definition.

$$A_0 = \phi \quad , \{A^i\} = \vec{A} \quad \Rightarrow \quad A_i = - A_i$$

$$E_i = - \nabla_i \phi - \partial_t A_i$$
$$= - \partial_i A_0 + \partial_0 A_i = F_{0i}$$

$$B_i = \epsilon_{ijk} \partial_j A_k$$
$$= - \epsilon_{ijk} \partial_j A_k$$
$$= -\frac{1}{2} \epsilon_{ijk} (\partial_j A_k - \partial_k A_j)$$

$$\Rightarrow \quad \epsilon_{ilm} B_i = -\frac{1}{2} \epsilon_{ilm} \epsilon_{ijk} (\partial_j A_k - \partial_k A_j)$$
$$= -\frac{1}{2} (\delta_{lj} \delta_{km} - \delta_{lk} \delta_{jm}) (\partial_j A_k - \partial_k A_j)$$
$$= -\frac{1}{2} (\partial_l A_m - \partial_m A_l - \partial_m A_l + \partial_l A_m)$$
$$= - (\partial_l A_m - \partial_m A_l)$$
$$\Rightarrow \quad (\partial_l A_m - \partial_m A_l) = - \epsilon_{ilm} B_i = F_{lm}$$

2nd eqn

$$- \partial_0 F_{0i} + \partial_j (\partial_j A_i - \partial_i A_j) = 0$$
$$\Rightarrow - \frac{\partial E_i}{\partial t} + \partial_j (- \epsilon_{kji}) B_k = 0$$
$$\Rightarrow + \frac{\partial E_i}{\partial t} + \epsilon_{ikj} \partial_j B_k = 0$$
$$\Rightarrow \boxed{\frac{\partial E_i}{\partial t} = \epsilon_{ijk} \partial_j B_k}$$