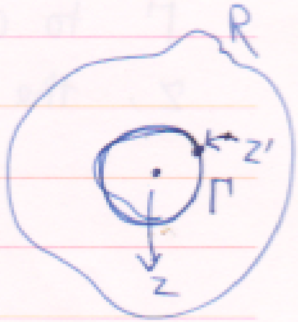
<u>Proof</u> : Let us consider the func^n

$$F(z,z') = \frac{f(z') - f(z)}{z' - z}$$

$f(z)$ is continuous in R means

$$|f(z') - f(z)| < \epsilon$$

whenever $|z' - z| < \delta(\epsilon)$



Let Γ be a circle in R centered at z and of radious $r < \delta(\epsilon)$.

Suppose we integrate along Γ

$$z' - z = re^{i\theta} \Rightarrow z' = z + re^{i\theta} \quad 0 \leq \theta \leq 2\pi$$

$$\Rightarrow \quad \left| \frac{f(z') - f(z)}{z' - z} \right| < \frac{\epsilon}{r} \quad \text{for } z' \text{ on } \Gamma$$

Therefore using Darboux inequality

$$\left| \int_\Gamma \frac{f(z') - f(z)}{z' - z} dz \right| \leq \frac{\epsilon}{r} 2\pi r = 2\pi\epsilon$$

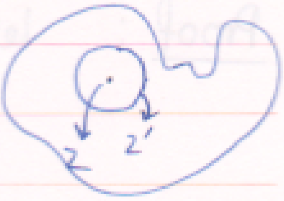
In the limit $\epsilon \to 0$ RHS tends to zero, hence

$$\int_\Gamma \frac{f(z') - f(z)}{z' - z} dz = 0$$

$$\Rightarrow \oint_\Gamma \frac{f(z')}{z' - z} dz' = f(z) \oint_C \frac{dz'}{z' - z} = f(z) \oint \frac{r i e^{i\theta} d\theta}{r e^{i\theta}}$$

$$= 2\pi i f(z)$$

$$\oint_\Gamma \frac{f(z')}{z'-z}\,dz' = 2\pi i\, f(z)$$



$\frac{f(z')}{z'-z}$ is an analytic func.^{n} of z' except at $z' = z$.

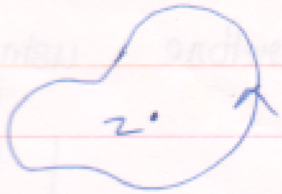
Therefore we can deform the circular contour Γ to any arbitrary contour C without touching z, the result will remain same. Hence,

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}\,dz'}$$ when z is interior to C.

If z is exterior to C, then $\frac{f(z')}{z'-z}$ is analytic on C and also interior to C, thus from Cauchy's theorem

$$\frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}\,dz' = 0$$ when z is exterior to C.

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(z')}{z'-z}\,dz'$$



$\downarrow$

This formula expresses completely the value of an analytic func.^{n} at any point z within the contour C, once its value on the boundary curve C is specified.

A very powerful result !!

Consider

$$f(z) = \frac{1}{2\pi i} \int_C \frac{g(z')}{z'-z} dz'$$

Where C is piecewise regular contour not necessarily closed. The contour has finite length. $g(z)$ is a function that is continuous on C.

Let us consider,

$$\Delta = \left| \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{g(z')}{(z'-z)^2} dz' \right|$$

$$= \left| \frac{1}{\Delta z} \left[\frac{1}{2\pi i} \int_C \frac{g(z')}{z'-z-\Delta z} dz' - \frac{1}{2\pi i} \int_C \frac{g(z')}{z'-z} dz' \right] - \frac{1}{2\pi i} \int \frac{g(z')}{(z'-z)^2} dz' \right|$$

$$= \left| \frac{1}{\Delta z} \frac{1}{2\pi i} \int_C dz' g(z') \frac{\cancel{z'} - \cancel{z} - \cancel{z'} + \cancel{z} + \Delta z}{(z'-z-\Delta z)(z'-z)} - \frac{1}{2\pi i} \int_C \frac{g(z')}{(z'-z)^2} dz' \right|$$

$$= \left| \frac{1}{2\pi i} \int_C dz' g(z') \left\{ \frac{\cancel{z'} - \cancel{z} - \cancel{z'} + \cancel{z} + \Delta z}{(z'-z-\Delta z)(z'-z)^2} \right\} \right|$$

$$= \left| \frac{\Delta z}{2\pi i} \int \frac{g(z')}{(z'-z-\Delta z)(z'-z)^2} dz' \right|$$

Since z is not on the contour, $z \neq z'$.
Therefore $\Delta \to 0$ as $\Delta z \to 0$

$$\Rightarrow \frac{df}{dz} = \frac{1}{2\pi i}\int_C \frac{g(z')}{(z'-z)^2}\,dz'$$

Ill^ly we can prove that,

$$\frac{d^n f}{dz^n} = \frac{n!}{2\pi i}\int_C \frac{g(z')}{(z'-z)^{n+1}}\,dz' \qquad \text{Ex: 2 Prove this.}$$

Now if $f(z)$ is an analytic func^n in some region R in the complex plane

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')}{z'-z}\,dz' \qquad z \notin C.$$

$$\Rightarrow \frac{d^n f}{dz^n} = \frac{n!}{2\pi i}\oint_C \frac{f(z')}{(z'-z)^{n+1}}\,dz'$$

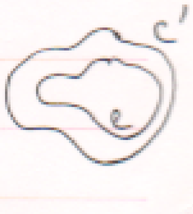
$\longrightarrow$ The derivative of all order of an analytic func^n are themselves analytic.

$$g(z) = \frac{df}{dz} = \frac{1}{2\pi i}\oint \frac{f(z')}{(z'-z)^2}\,dz' \; \cancel{= \frac{1}{2\pi i}\oint f(z) + (z'-z)\left.\frac{df}{dz'}\right|_{z'=z}}$$

is analytic

$$\int_C g(z)\,dz = \frac{1}{2\pi i}\oint_{C'} dz'\, f(z') \oint_C \frac{dz}{(z'-z)^2} \qquad C \text{ should be inside } C'$$

$\frac{1}{(z'-z)^2}$ is analytic inside C and on C



$$\boxed{\oint_C g(z)\,dz = 0}$$

Morera's theorem

This is converse of Cauchy's theorem.

* If the integral $\int_C f(z)\,dz$ of a func^n, which is continuous in some region, vanishes for any closed contour C lying within this region, then $f(z)$ is analytic in that region.

Proof: Home Work.

Taylor Series

Theorem: Let $f(z)$ be a func^n, analytic within and on a circle Γ centered at $z = z_0$. The value of this function at any point z within Γ is given by the uniformly convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Where, $$a_n = \frac{1}{n!} \frac{d^n f}{dz^n}\bigg|_{z=z_0} = \frac{1}{2\pi i} \int_\Gamma \frac{f(z')}{(z'-z_0)^{n+1}}\, dz'$$

Proof: Since $f(z)$ is analytic

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z')}{z'-z}\, dz' = \frac{1}{2\pi i} \oint \frac{f(z')}{(z'-z_0)} \frac{1}{\left(1 - \frac{z-z_0}{z'-z_0}\right)}$$

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z')}{(z'-z_0)} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{z'-z^0} \right)^n$$

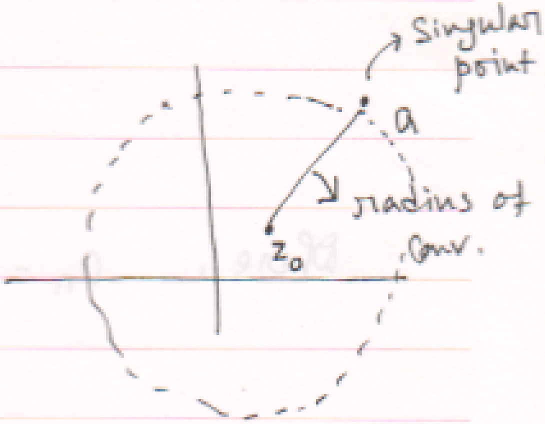
$$= \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z')}{(z'-z_0)^{n+1}}$$

The above series is convergent since $\left| \frac{z-z_0}{z'-z_0} \right| < 1$.

z' is a point on Γ and z is a point inside Γ.

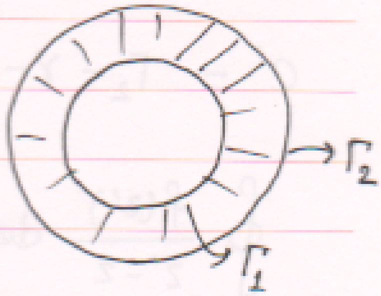
Radius of convergence of Taylor series :

Cauchy's formula breaks down when Γ goes throug or includes is point of singularity of $f(z)$. Therefore radius of convergent can not be greater than the distance between the point $z=z_0$ and the nearest singularity of $f(z)$.



The Laurent Series

Suppose the funcⁿ $f(z)$ is not analytic throughout the whole interior of a circle (what was the case for Taylor series). But it is analytic between the annular region between two circles Γ_1 and Γ_2. In such case the funcⁿ can also be expanded about the center of the two circles $\longrightarrow$ Laurent expansion.



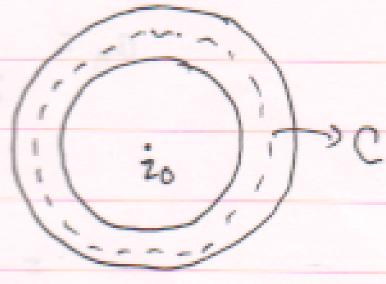
Theorem: Let $f(z)$ be analytic in the annular region between and on two concentric circles Γ_1 and Γ_2 centered at $z = z_0$. The value of $f(z)$ at any point z within the annular region is given by the uniformly convergent power series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where,

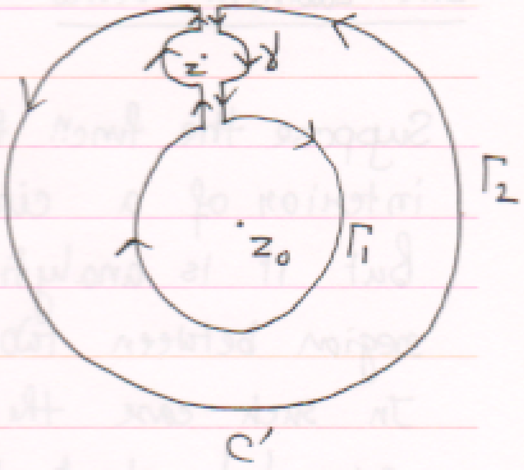
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z')}{(z'-z_0)^{n+1}}\, dz'$$

C is a contour in the annular region encircling the point z_0.



Proof:

Draw a small contour γ around z and in annular region.



$$C' = \Gamma_2 - \gamma - \Gamma_1$$

$$\underbrace{\oint_{C'} \frac{f(z')}{z'-z}\,dz'}_{= 0\ \cancel{2\pi i f(z)}} = \oint_{\Gamma_2} \frac{f(z')}{z'-z}\,dz' - \oint_{\gamma} \frac{f(z')}{z'-z}\,dz' - \oint_{\Gamma_1} \frac{f(z')}{z'-z}\,dz'$$

$$\Rightarrow \underbrace{\oint_{\gamma} \frac{f(z')}{z'-z}\,dz'}_{= 2\pi i f(z)} = \oint_{\Gamma_2} \frac{f(z')}{z'-z}\,dz' - \oint_{\Gamma_1} \frac{f(z')}{z'-z}\,dz'$$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f(z')}{z'-z}\,dz' - \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(z')}{z'-z}\,dz'$$

$$\frac{1}{2\pi i}\oint \frac{f(z')}{z'-z}\,dz' = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$

$$= \frac{1}{2\pi i}\oint_{C} \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$ Since $\frac{f(z')}{(z'-z_0)^{n+1}}$ is analytic.

For the second term : $-\frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(z')}{z'-z}\, dz'$

$$(z'-z) = (z'-z_0) + (z_0 - z) = -(z-z_0) + (z'-z_0)$$

$$= -(z-z_0)\left[1 - \frac{z'-z_0}{z-z_0}\right]$$

$$\frac{1}{z'-z} = -\frac{1}{(z-z_0)}\left[1 - \frac{z'-z_0}{z-z_0}\right]^{-1}$$

$$= -\frac{1}{z-z_0}\left[\sum_{n=1}^{\infty}\left(\frac{z'-z_0}{z-z_0}\right)^{n-1}\right] \rightarrow \text{Convergent since } \left|\frac{z'-z_0}{z-z_0}\right| < 1.$$

$$\Rightarrow -\frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(z')}{z'-z}\, dz' = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(z')}{(z-z_0)^n} \sum_{n=1}^{\infty} (z'-z_0)^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad b_n = \frac{1}{2\pi i}\oint_{\Gamma_1} (z'-z_0)^{n-1} f(z')\, dz'$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

Combining thes two results

$$f(z) = \sum_{n=-\infty}^{\infty} d_n (z-z_0)^n \qquad d_n = \oint_C \frac{f(z')}{(z'-z_0)^{n+1}}\, dz'$$

C be a contour in annular region and encircling the point z_0.