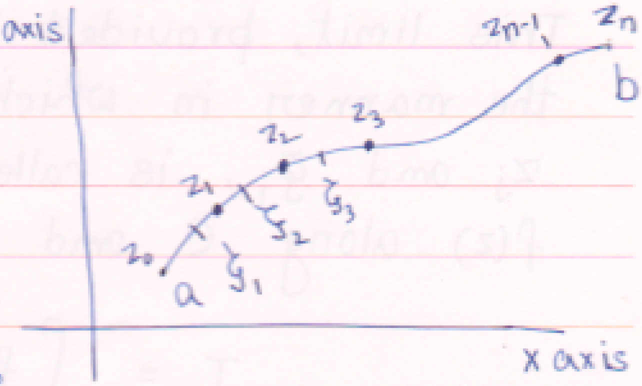
Integral Calculus of a function of complex variable.

Let $f(z)$ be a funcⁿ and C be a curve in the complex plane from a to b.



Curve C is regular.
Regular means it can be described by a parametric eqⁿ

$$z = z(t) \equiv x(t) + i\,y(t) \qquad t_a \le t \le t_b$$

$$z(t_a) = a, \quad z(t_b) = b.$$

t is a real parameter, $x(t)$ and $y(t)$ are real single valued funⁿs and they have continuous first order derivatives.

The discussion is also valid for a piecewise regular curve: A continuous curve consisting of finite no. of regular arcs.

Subdivide ab into n intervals introducing n+1 points $z_0, z_1, z_2 \ldots z_n$. $z_0 = a$, $z_n = b$.

Lets take n points $\xi_1, \xi_2, \cdots, \xi_n$ on C such that ξ_k lies between z_k and z_{k-1}

Define the sum: $I_n = \sum_{k=1}^{n} f(\xi_k)\,(z_k - z_{k-1})$

Take the limit $n \to \infty$ in such a manner that

$$|z_k - z_{k-1}| \to 0 \quad \forall\, k.$$

This limit, provided it exists and is independent of the manner in which we have chosen the points z_j and ξ_j, is called the contour integral of $f(z)$ along C and is written as,

$$I = \int_C f(z)\, dz$$

contour integration depends on the choice of the contour.

Since $f(z) = u + iv$ and $dz = dx + i\,dy$

$$I = \int_C (u + iv)(dx + i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int (v\,dx + u\,dy)$$

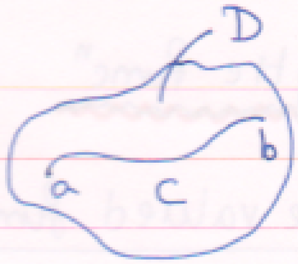
If the curve is parametrized by t

Then $$I = \int_{t_a}^{t_b} \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right)dt + i\int_{t_a}^{t_b}\left(v\frac{dx}{dt} + u\frac{dy}{dt}\right)dt$$

Since $\frac{dx}{dt} + i\frac{dy}{dt} = \frac{dz}{dt}$

$$I = \int_{t_a}^{t_b} (u + iv)\frac{dz}{dt}\,dt = \int_{t_a}^{t_b} f[z(t)]\frac{dz}{dt}\,dt$$

Suppose in the domain D (C belongs to D) $f(z)$ can be written as



$$f(z) = \frac{dF(z)}{dz}$$

$F(z) \rightarrow$ primitive func^n. of $f(z)$.

$$I = \int_{t_a}^{t_b} f[z(t)] \frac{dz}{dt}\, dt$$

$$= \int_{t_a}^{t_b} \frac{dF(z(t))}{dt}\, dt = F(b) - F(a)$$

I does not depend on the contour C but only end points a and b.

Upper bound of ~~complex~~ contour integral

$$I = \int_C f(z)\, dz$$

Assume that $|f(z)|$ bounded on C (finite on C).

$$I = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \sum_{k=1}^{n} (z_k - z_{k-1}) f(\xi_k)$$

$$|I_n| \leq \sum_{n=1}^{n} |z_k - z_{k-1}| \, |f(\xi_k)|$$

$$\leq \max|f| \sum_{n=1}^{n} |z_k - z_{k-1}|$$

max$|f|$ $\rightarrow$ maximum modulus of f on C.

$$\leq \max|f| \cdot \underbrace{L}_{\text{length of the contour}}$$

In $n \to \infty$ limit

$$\boxed{\left| \int_C f(z)\, dz \right| \leq \max|f| \cdot L}$$

Darboux's Inequality.