Ex: 1 Show that

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ , \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

A funcn of n variables $\{x_1, x_2, \ldots, x_n\}$ satisfying

$$\sum_{j=1}^{n} \frac{\partial^2 h}{\partial x_j^2} = 0 \longrightarrow \text{Laplace eqn.}$$

is called harmonic function.

Thus real and imaginary parts of a complex funcn separately satisfy the Laplace equation and are harmonic funcn of two variables.

Important: Converse is not true. A pair of harmonic funcns does not in general define a differentiable funcn.

Example: $f(z) = x + 2iy$

The Cauchy-Riemann conditions we have derived are necessary conditions. Because we have choosen only two specific limiting process. Therefore, the question is if the ~~Ca~~ CR conditions are sufficient for $f(z)$ to be differentiable.

Theorem : Let the real and imaginary parts $u(x,y)$ and $v(x,y)$ of a funcn of a complex variable $f(z)$ obey CR equations and also posses continuous first partial derivatives with respect to the two variables x and y at all points of some region of the complex plane. Then $f(z)$ is differentiable throughout this region.

Proof : See book.

Implications of CR conditions - A

$$f(z) = u(x,y) + i\,v(x,y)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{n!}{k!\,(n-k)!} (x-x_0)^{n-k} (y-y_0)^{k} \times \frac{\partial^n}{\partial^{n-k} x \, \partial^k y} \Big[u(x,y) + i\,v(x,y) \Big]_{\substack{x=x_0 \\ y=y_0}}$$

From CR conditions we get

$$\frac{\partial}{\partial y}\Big[u+iv\Big]_{(x_0,y_0)} = i\,\frac{\partial}{\partial x}\Big[u+iv\Big]_{(x_0,y_0)}$$

$$\Rightarrow \frac{\partial^k}{\partial y^k}\Big[u+iv\Big]_{(x_0,y_0)} = i^k \frac{\partial^k}{\partial x^k}\Big[u+iv\Big]_{(x_0,y_0)}$$

Then,

$$f(z) = u(x,y) + i v(x,y)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{n!}{k!\,(n-k)!} (x-x_0)^{n-k} (y-y_0)^{k} \, i^{k} \frac{\partial^n}{\partial x^n} \Big[u(x,y) + i v(x,y) \Big]_{(x_0, y_0)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ (x+iy) - (x_0+iy_0) \right\}^n \frac{\partial^n}{\partial x^n} \Big[u(x,y) + i v(x,y) \Big]_{(x_0, y_0)}$$

Due to the CR conditions, the two real variables x and y enter into the func^n in the unique combinations $(x+iy)$.

Thus CR conditions have a consequence that a ~~mathe~~ differentiable func^n depends only on the combination $z = x+iy$, not any other combination $\bar{z} = x-iy$.

However, the above conclusion depends on the assumption that the func^n can be expanded in a Taylor Series.

<u>Important</u>: An expression only depends on z and not on $\bar{z}$ does not ensure the differentiability. Ex: $\text{Im}\, z$ is not differentiable at $z=0$. [Because $\text{Im}\, z$ can not be expanded about $z=0$]

Any function which is differentiable ~~at a point~~ within a neighbourhood of a point can be expanded in a Taylor series about that point. Therefore the expression that defines such a func^n can explicitly depend on z.

Consider the example $f(z) = x + 2iy$. $f(z)$ was not differentiable at $z=0$. Because $f(z)$ can not be written as a func^n of z only.

$$f(z) = \frac{3}{2} z - \frac{1}{2} \bar{z}$$

Implications of CR conditions - B

Let $f(z) = u(x,y) + i v(x,y)$ is differentiable at (x_0, y_0)

$$\vec{\nabla} u = \hat{x} \frac{\partial u}{\partial x} + \hat{y} \frac{\partial u}{\partial y} \; ; \quad \vec{\nabla} v = \hat{x} \frac{\partial v}{\partial x} + \hat{y} \frac{\partial v}{\partial y}$$

$$\Rightarrow (\vec{\nabla} u) \cdot (\vec{\nabla} v) \Big|_{(x_0, y_0)} = \left[\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \right]_{(x_0, y_0)} = 0 \quad [\text{using CR}]$$

Now consider two curves passing through (x_0, y_0)

$$u(x,y) = u(x_0, y_0) = C_1$$

$$\text{and} \quad v(x,y) = v(x_0, y_0) = C_2$$

Now tangent vectors to these curves at (x_0, y_0) are $\vec{\nabla} u$ and $\vec{\nabla} v$. Since $(\vec{\nabla} u \cdot \vec{\nabla} v) = 0$ at (x_0, y_0) we conclude: The tangent to the curves $\mathrm{Re}\, f(z) = \mathrm{Re}\, f(z_0)$ and $\mathrm{Im}\, f(z) = \mathrm{Im}\, f(z_0)$ are perpendicular ~~to~~ at the point z_0 if $f(z)$ is differentiable in the neighbourhood of this point.