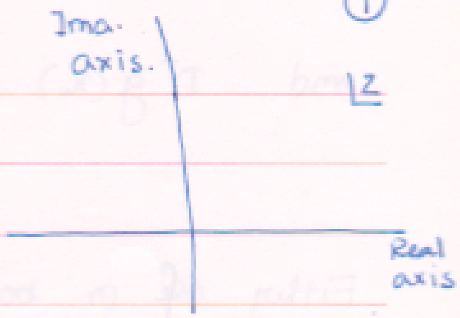
Complex number $z = x + iy$ $\quad i = \sqrt{-1}$

↓

Denoted by a poin in complex z plane.



~~Complex~~ $\boxed{\text{Functions of complex argument:}}$

$f(z) \rightarrow$ Defined over a set of complex number

$\equiv$ Defined over a set of points in a plane.

$f(z)$ is a function of two real variables.

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

$$\boxed{f(z) = \underbrace{u(x,y) + i\,v(x,y)}_{\text{Real functions.}}}$$

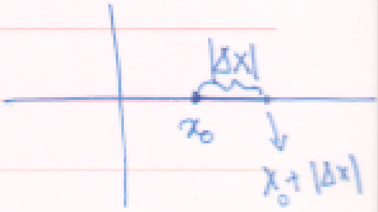
→ Theory of func^n of complex variables reduces to theory of func^{n}s of two real variables.

We shall deal with a very specific set of func^{n}s : analytic func^{n}s. We shall gradually define what analytic func^n is. Let us start of differentiation of a func^n of complex argument.

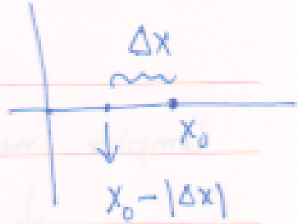
Differentiability : Suppose $g(x)$ func^n of a real variable x. We want to check if the func^n is differentiable ~~at~~ in the neighbourhood of the point $x = x_0$.

Consider the following two limits:

$$D^+ g(x_0) = \underset{|\Delta x| \to 0}{\mathrm{Lt}} \frac{g(x_0 + |\Delta x|) - g(x_0)}{|\Delta x|}$$



and $\quad D^{-} g(x_0) = \underset{|\Delta x| \to 0}{\text{Lt}} \dfrac{g(x_0 - |\Delta x|) - g(x_0)}{-|\Delta x|}$

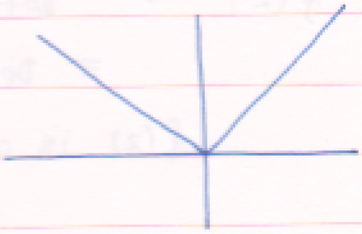


Either of or both limits may not exists at all, even though $g(x)$ may be continuous. $\quad g(x) = \sqrt{1-x^2}$

It may also happen that the two limits do exists but they are different.

At $x = 0 \quad D^{+} g(0) = +1$

$\qquad\qquad D^{-} g(0) = -1$



Any func^n of a sharp turning point : $D^{+} \neq D^{-}$.

* When the two limits finite and equal $D^{+} g(x_0) = D^{-} g(x_0)$ the func $g(x)$ is said to be differentiable at $x = x_0$ and the limit

$$\lim_{\Delta x \to 0} \frac{g(x_0 + \Delta x) - g(x)}{\Delta x}$$

is called derivative of $g(x)$ at $x = x_0$ and denoted by

$$\left.\frac{dg(x)}{dx}\right|_{x=x_0}$$

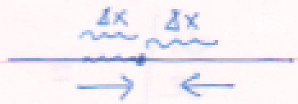
Derivative is a local concept or characteristic. It tells the behaviour of the func^n at a particular point.

Differential calculus of function of complex variables

The derivative of a func^n of complex variable w.r.t. its argument z is formally defined in the same way as it for a func^n of real variables

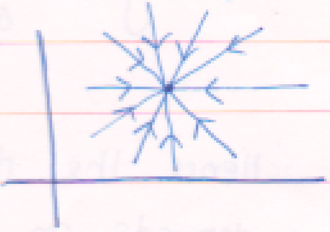
$$\boxed{\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}}$$

But taking the limit $\Delta z \to 0$ is crucial here. One a real line there is only two possibilities. Because taking $\Delta x \to 0$ means how you are approaching the corresponding point. For real line there are only two possibilities.



But $\Delta z = \Delta x + i\Delta y$. Thaking $\Delta z \to 0$ means one needs to take both Δx and $\Delta y \to 0$. There are ∞ no. of possibilities to take the limit.

$$\Delta z = |\Delta z|\, e^{i\theta}$$



Now $\Delta z \to 0$ for $|\Delta z| \to 0$ for any θ.
But the derivative will depend on the argument θ. Or in term of Δx and Δy the limit depends on the order in which ~~we are~~ $\Delta x, \Delta y$ tend to zero.

For example: $f(z) = x + i2y$ calculate $f'(z)$ at the Origin.

$$\left.\frac{df}{dz}\right|_{z=0} = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x + i2\Delta y}{\Delta x + i\Delta y}$$

~~Therefore~~ $\frac{df}{dz}$ ~~at~~ $z=0$ depends on the order in which we take $\Delta x, \Delta y \to 0$

i) ~~Δx~~ Δx is fixed take $\Delta y \to 0$ first

$$\left.\frac{df}{dz}\right|_{z=0} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x + i2\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

ii) Δy is fixed, $\Delta x \longrightarrow 0$ first.

$$\left.\frac{df}{dz}\right|_{z=0} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x + i2\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y \to 0} \frac{2i\Delta y}{i\Delta y} = 2$$

iii) Take $\Delta x, \Delta y \to 0$ along $y = \alpha x$

$$\frac{df}{dy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x + i2\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \to 0} \frac{\Delta x + i2\alpha\Delta x}{\Delta x + i\alpha\Delta x}$$

$$= \frac{1+2\alpha i}{1+i\alpha}$$

Hence the derivative depends on the argument of $\Delta z = \frac{\Delta y}{\Delta x} = \alpha$.

Therefore we define a function $f(z)$ is differentiable at a point $z = z_0$ if the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

exists, is finite and does not depend on the manner in which one takes the limit. Or in other words does not depend on ~~the~~ the way one approaches the point $z = z_0$.

Thus differentiability of a func^n of complex argument is much more restrictive than a fun^r of real variable. For a func^n of real variable one can approach a point in two different ways, either from left or from right. Where as a point in complex plane can be approached in infinite no. of ways. Therefor differentiability of a func^n of real variable depends only one one requirement ~~but~~ ~~for a~~ whether D^+ and D^- are same. But for a func^n of complex argument we need to impose an infinite no. of requirements.

Rules of differentiation : Same as in the real variable case.

Cauchy - Riemann conditions

$$f(z) = u(x,y) + i\,v(x,y)$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + i\left\{v(x+\Delta x, y+\Delta y) - v(x,y)\right\}}{\Delta x + i\,\Delta y}$$

If $f(z)$ is differentiable at $z \Rightarrow$ L.H.S is same for whatever order in which $\Delta x, \Delta y \to 0$ is taken.

First take $\Delta y = 0$ and $\Delta x \to 0$

$$\frac{df}{dz} = \frac{\partial u(x,y)}{\partial x} + i\,\frac{\partial v(x,y)}{\partial x} \qquad \text{———} \quad (1)$$

Then take $\Delta x = 0$ and $\Delta y \to 0$

$$\frac{df}{dz} = -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{———} \quad (2)$$

Since (1) and (2) are same by definition we get

$$\boxed{\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= \frac{\partial v(x,y)}{\partial y} \\ \frac{\partial u(x,y)}{\partial y} &= -\frac{\partial v(x,y)}{\partial x} \end{aligned}}$$

CAUCHY - RIEMANN CONDITIONS