Week 6

PHY 106 Quantum Physics
Instructor: Sebastian Wüster, IISER Bhopal, 2018

2.4) Wave properties of particles

Week4: (elm) waves are also particles!

(1924) de Broglie:
What if particle are also waves?

Motivation: Symmetry/ Aesthetics, not experiment!

Wave properties of particles

First questions:

What would be their wavelength?

What is the “medium”?
What is “waving”?

Let’s answer the first one in 2.4.1.)

2.4.1) Matter waves

Let’s try the same as for photons:

de-Broglie (matter) wave length

$$\lambda_{dB} = \frac{h}{p} \left(= \frac{h}{\gamma m v} \right) \quad (55)$$

- Here p is the momentum of the particle

- Can use non-relativistic *p=mv* if *v<<c*

- Also $p = \hbar k$

Matter waves

Let's also connect frequency and energy like for photons:

Matter wave frequency

$$\nu = \frac{E}{h} \quad (56)$$

- E is the **total** energy of the particle

- For example (free particle). $E = \frac{p^2}{2m}$ $\left(E = \gamma m c^2\right)$

Matter waves

Now we can already write a

Quantum wave function e.g.

$$\Psi(x) = A\cos[2\pi\,(\frac{x}{\lambda_{dB}} - \nu t)] \qquad (57)$$

- But what interpretation do we give the **amplitude** of this wave? What is **A**?

Matter waves

Quantum wave function

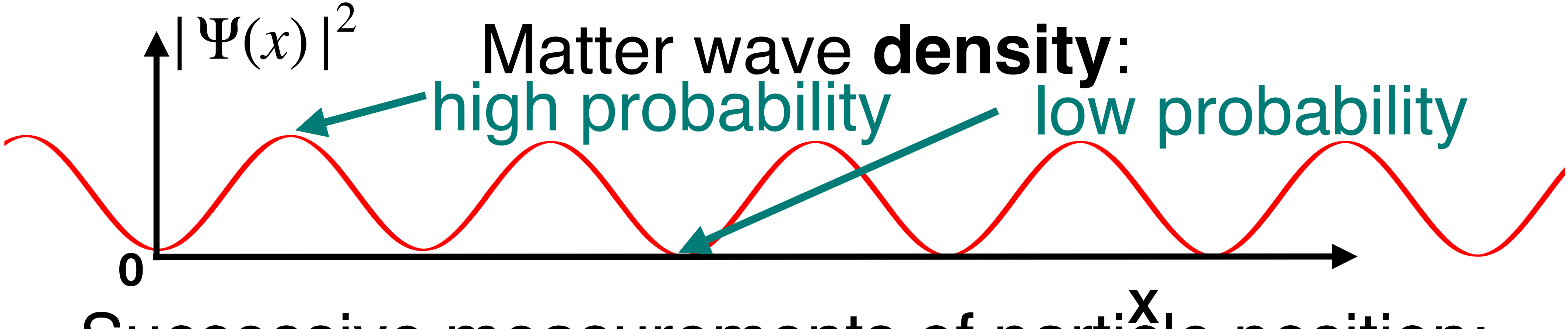
$$\Psi(x) = A\cos[2\pi\,(\frac{x}{\lambda_{dB}} - \nu t)] \qquad (57)$$

Turns out an interpretation describing the world is:

Probability for position

We find particle between x and x+dx with probability $\rho(x) = |\,\Psi(x)\,|^2 dx$ (58)

Matter wave **density**:



Successive measurements of particle position:

https://www.youtube.com/watch?v=hv12oB_uyFs

Matter waves

Quantum wave function e.g.

$$\Psi(x) = A\cos[2\pi\,(\frac{x}{\lambda_{dB}} - \nu t)]$$

repeat (57)

- $|\Psi(x)|^2$ is called **probability density**
- $Pr = \int_a^b |\Psi(x)|^2\,dx$ is the probability for the particle to be between locations *a* and *b*
- Thus *A* has units $m^{-d/2}$ in *d* dimensions.

Matter waves

Normalisation of wave function

$$1 = \int_{-\infty}^{\infty} |\Psi(x)|^2 dx \qquad (59)$$

- Particle has to be **somewhere** with *Pr=1.*
- $\tilde{f}(x)$ now satisfies Eq. (59), even if *f* didn't
- We can **normalise** most functions *f(x)* by

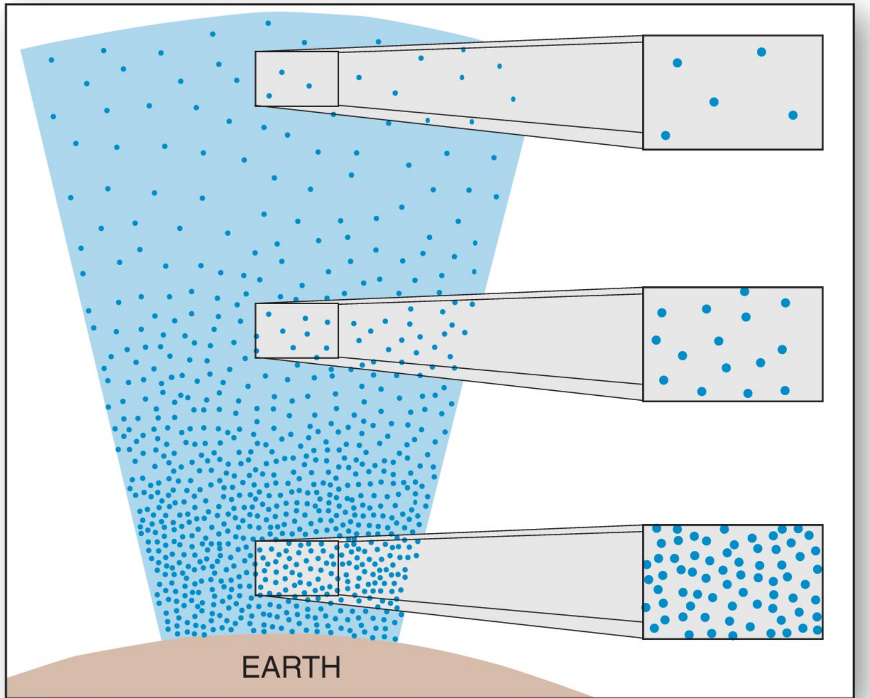
$$\tilde{f}(x) = \frac{f(x)}{\sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx}}$$

- **Note:** cos in Eq. (57) cannot be normalized if $-\infty < x < \infty$. Consider it within some box $-L < x < L$ only.

Examples: Probability distribution for the position of object can also arise classically

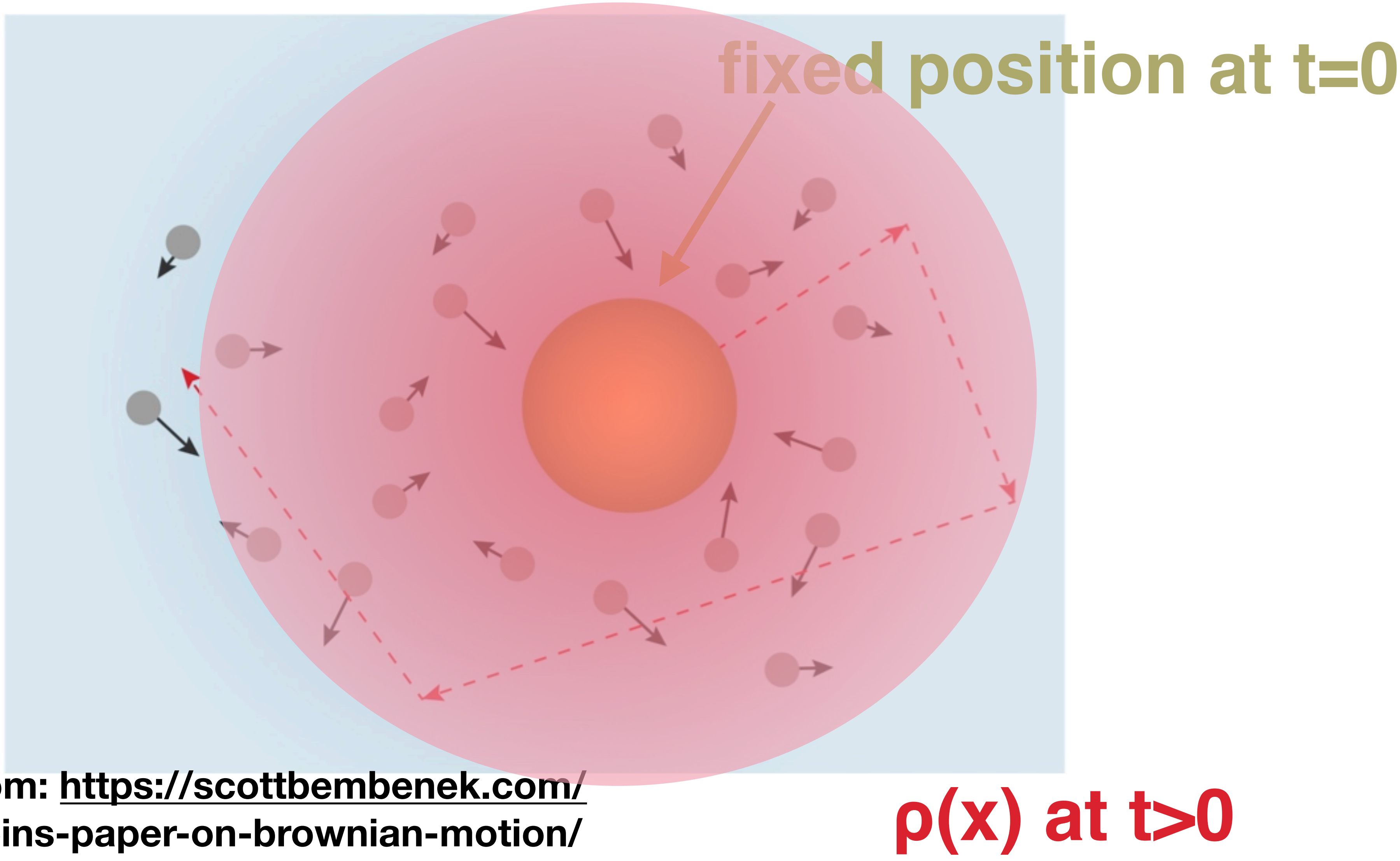
(1) Probability distribution for a random molecule in the atmosphere (~air density).

DRAW
p(h)



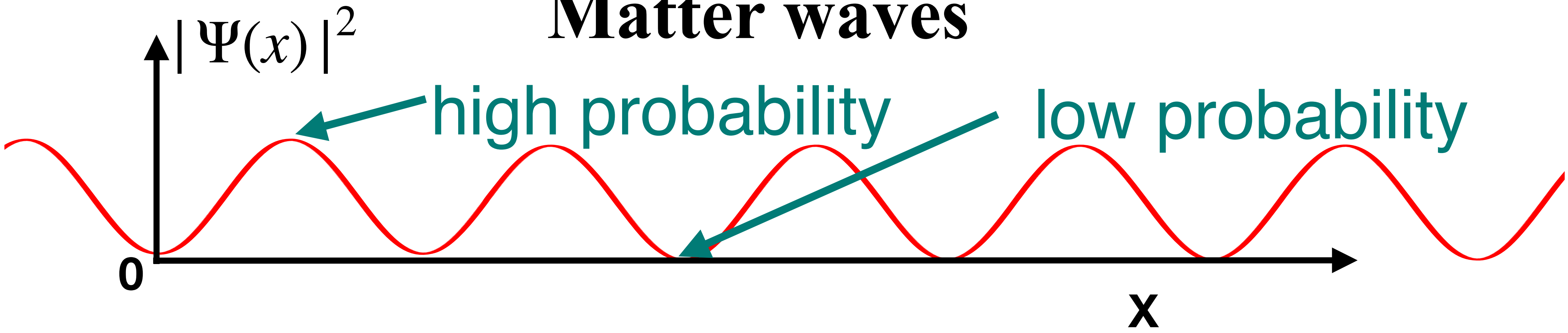
(we can't or don't need to know the position of each molecule) http://galileoandeinstein.physics.virginia.edu/more_stuff/Applets/Brownian/brownian.html

(2) In Brownian motion, particle in solution gets kicked around by solvent molecules…



Pic from: https://scottbembenek.com/einsteins-paper-on-brownian-motion/

Matter waves



Superposition state

Quantum mechanically the particle exists **at all** positions *x* with $p(x)>0$ **at once.** (60)

- Experiments show that an interpretation like a classical prob. dist. does **not work**

(for experimental demo, see electron-double slit example below)

Matter waves

Examples: De-Broglie wavelengths

electron $m_e = 9.10938356 \times 10^{-31}$ **kg**

let $v \approx 0.1c$

From Eq. (55): $$\lambda_{dB} = \frac{h}{mv} \approx 1.5\text{Å}$$

Unit: 1 Ånström = 1 Å = $1 \times 10^{-10} m$

"Size of atoms": ~1-5 Å, matter wave nature of electron **may be important**.

Matter waves

Examples: De-Broglie wavelengths

you: $m_{student} = 80$ **kg**

let $v \approx 1 \mu m/s$

From Eq. (55): $$\lambda_{dB} = \frac{h}{mv} \approx 5.2 \times 10^{-29} m$$

“Size of you”: ~1 m, matter wave nature of yourself **probably often un-important**.

What with *p=0?*

2.4.2) Evidence for Matter waves

Recall, beginning week 4:
Waves show **interference** and **diffraction**, particles do not.

After de Broglies proposal: Interference experiments with particles to test the idea

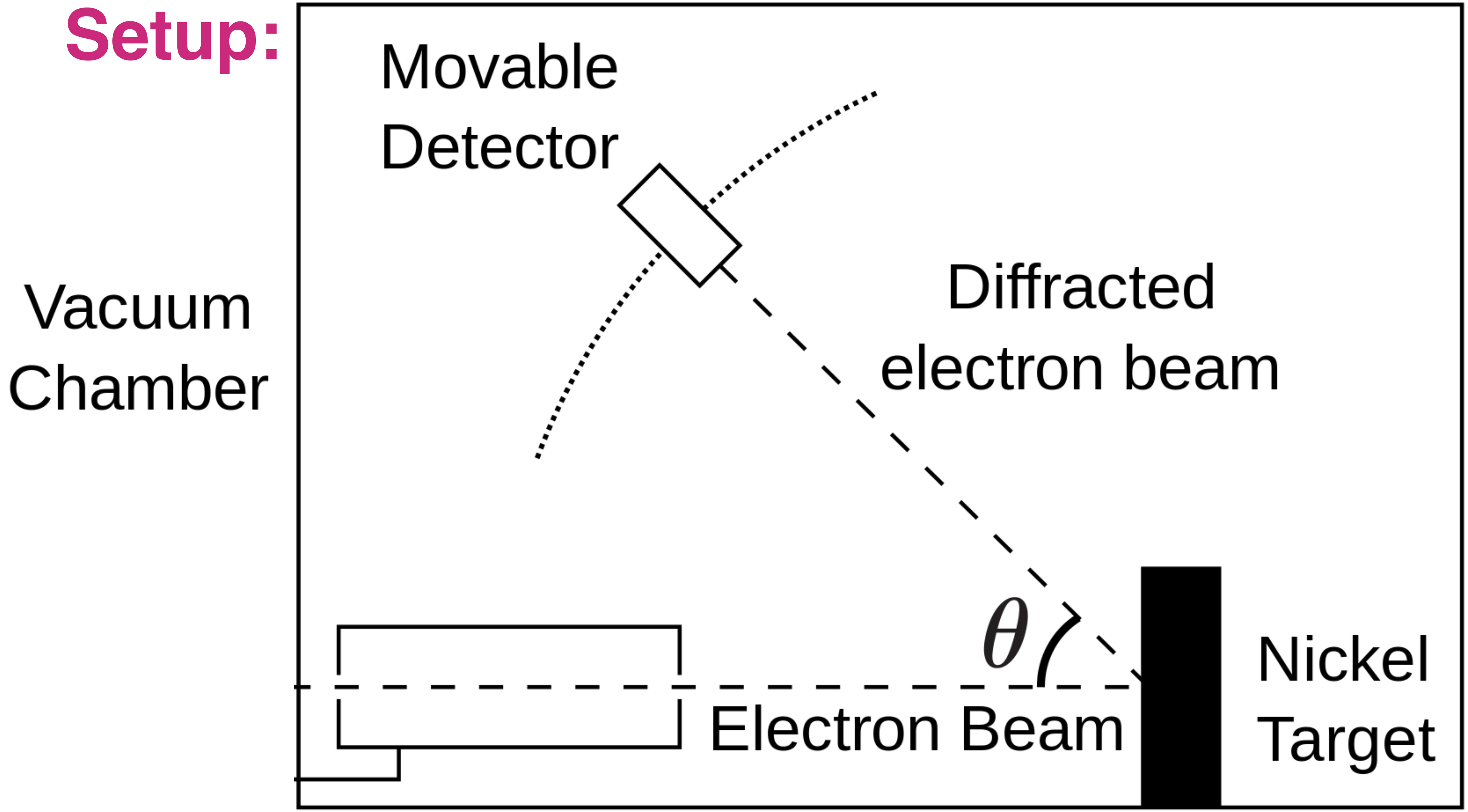
Problem: λ_{dB} very small, need short *d* for slits (2.1.4., week 3)

same solution as section 2.2.4), use solid metal crystal as **grating (slits)**

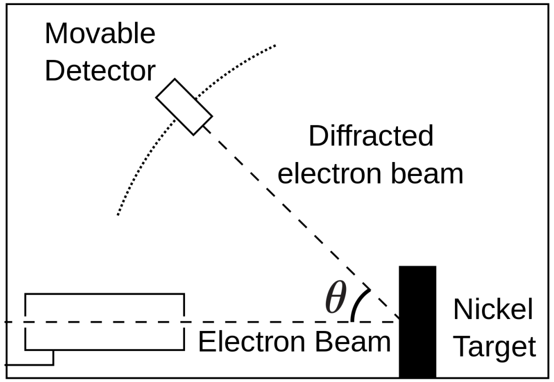
Evidence for Matter waves

Davisson - Germer experiment (1927)

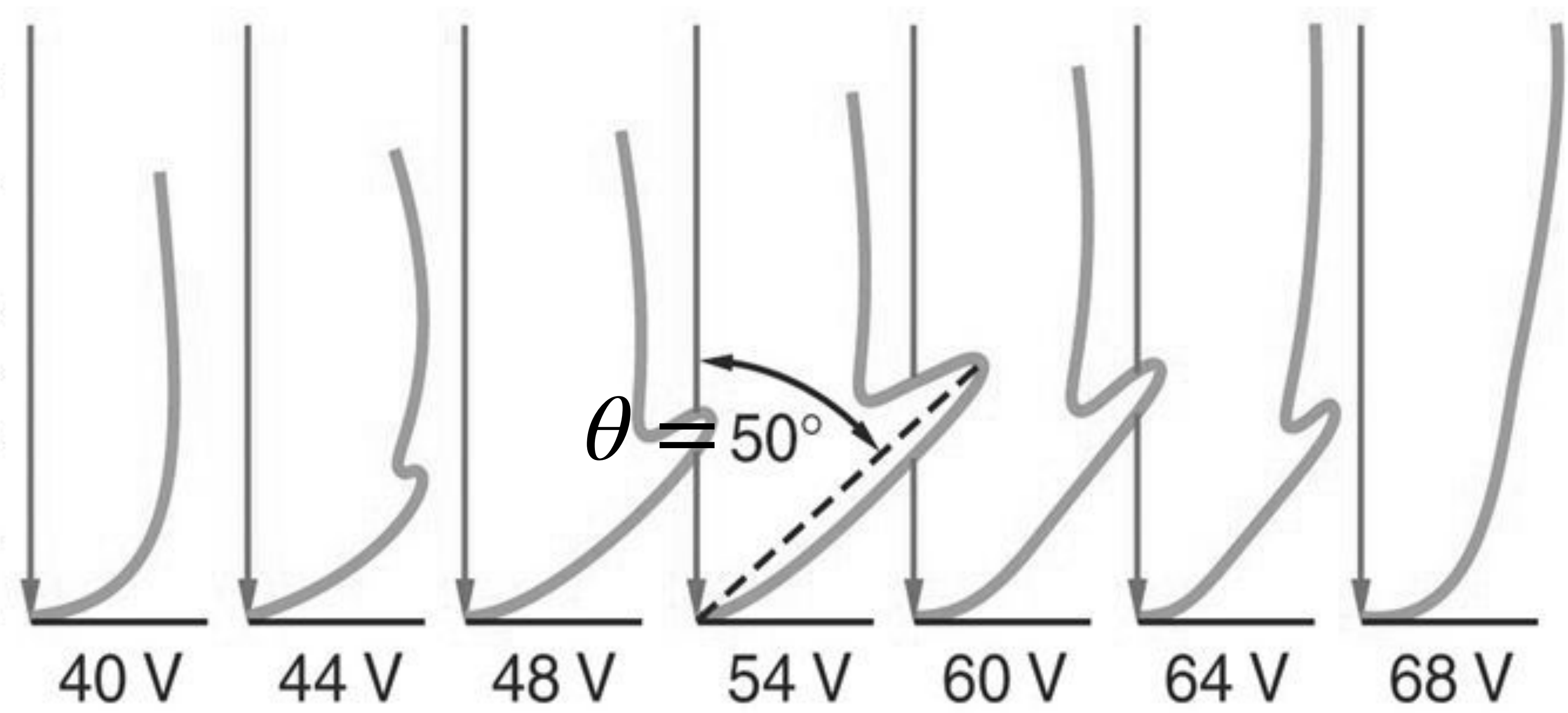
Setup:



Davisson - Germer experiment (1927)



Results: Number scattered electrons for different angles θ and acceleration voltages V.



Evidence for Matter waves

Analysis: Use **Bragg** scattering (section 2.2.4), works for any waves, also matter waves.

Eq. (32) $2d\sin(\theta) = n\lambda \longrightarrow \lambda = 0.165$ **nm**

Davisson-Germer: Use $\theta = 65^o \quad n = 1$, $d = 0.091$ **nm**

Electron $54\ \mathbf{eV} = E_{kin} = \frac{1}{2}mv^2 \rightarrow mv = p$

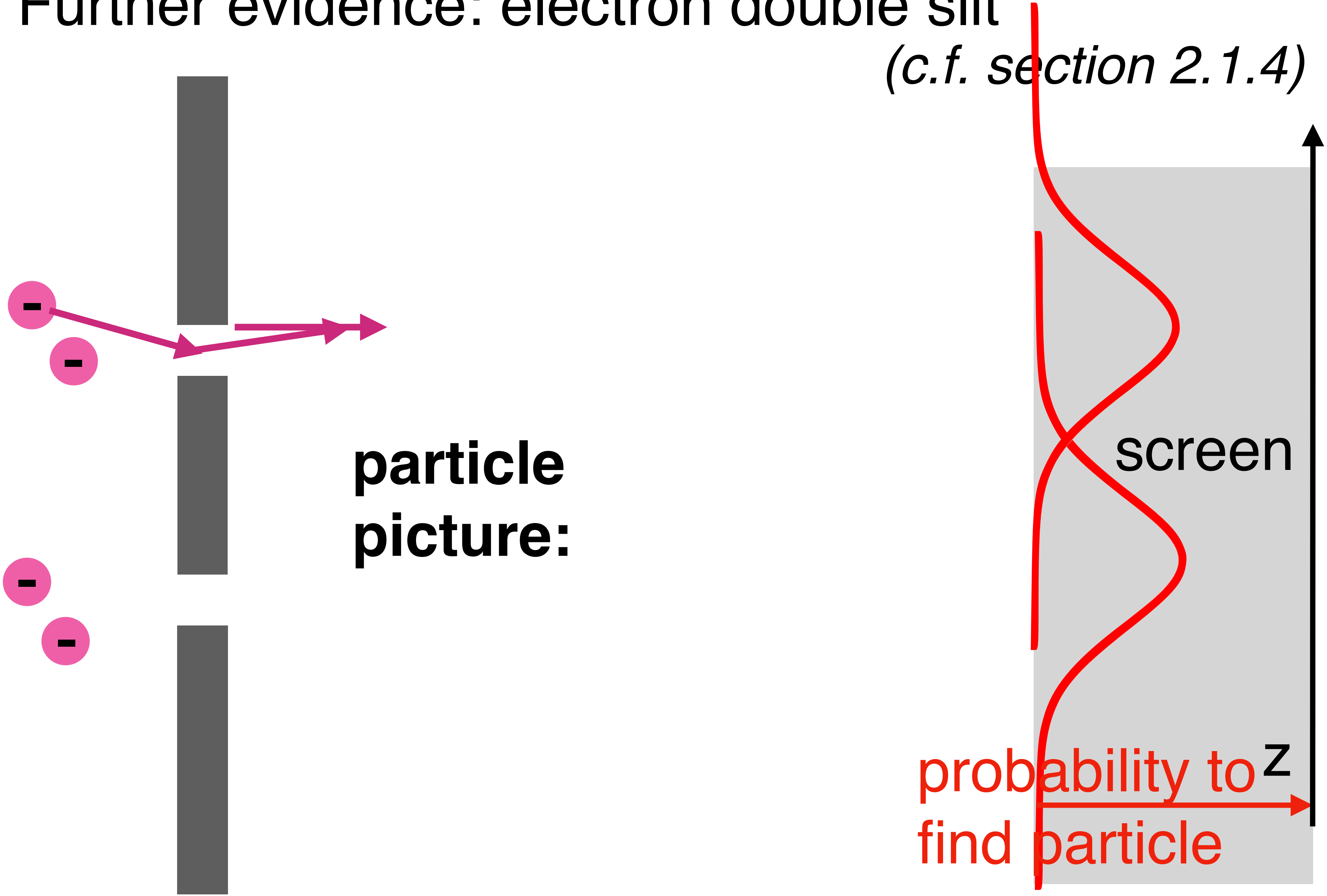
De-Broglie λ Eq. (55): $\lambda_{dB} = \frac{h}{mv} = 0.166$ **nm**

Conclusion: Electrons are matter waves!!!

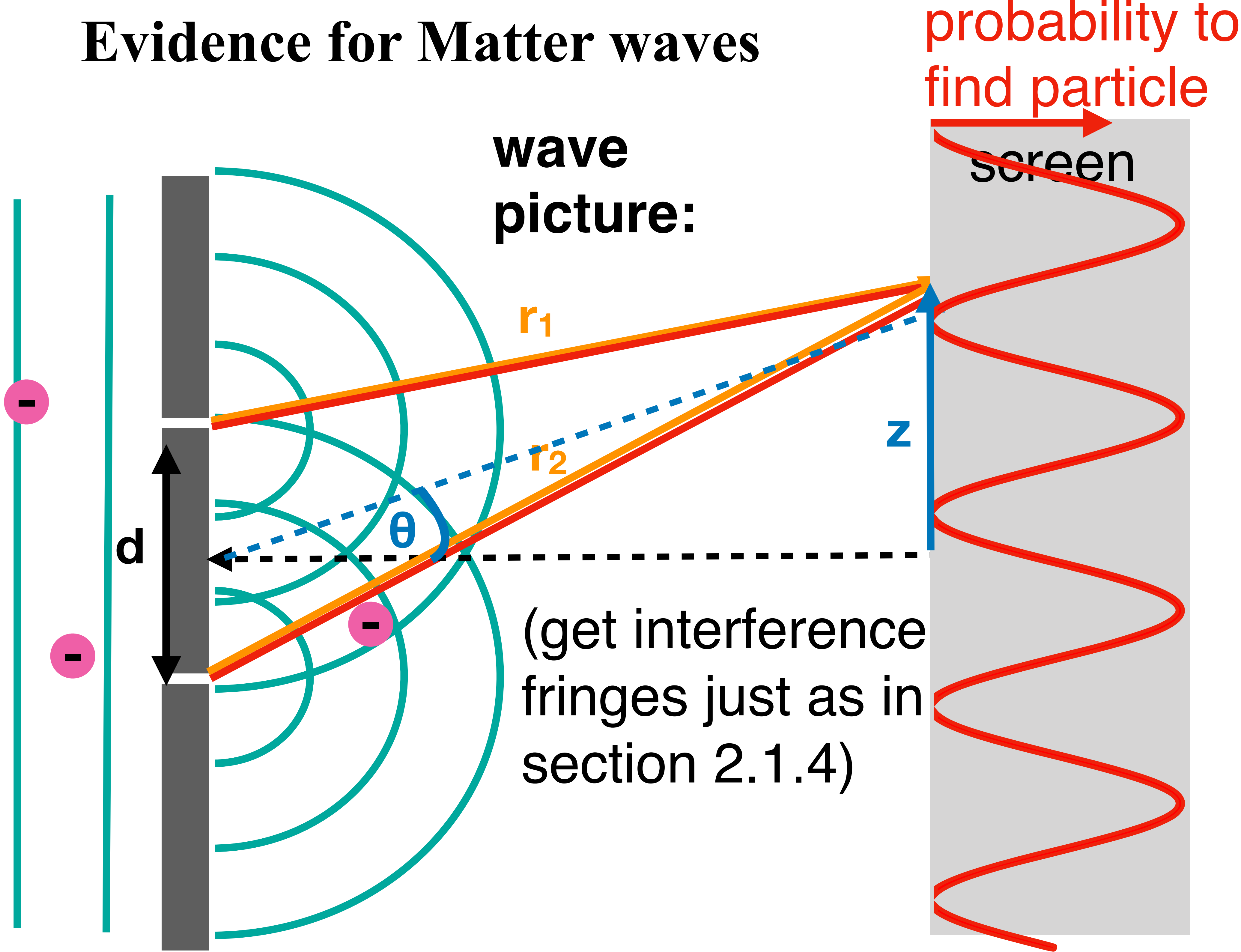
Evidence for Matter waves

Further evidence: electron double slit

(c.f. section 2.1.4)



Evidence for Matter waves

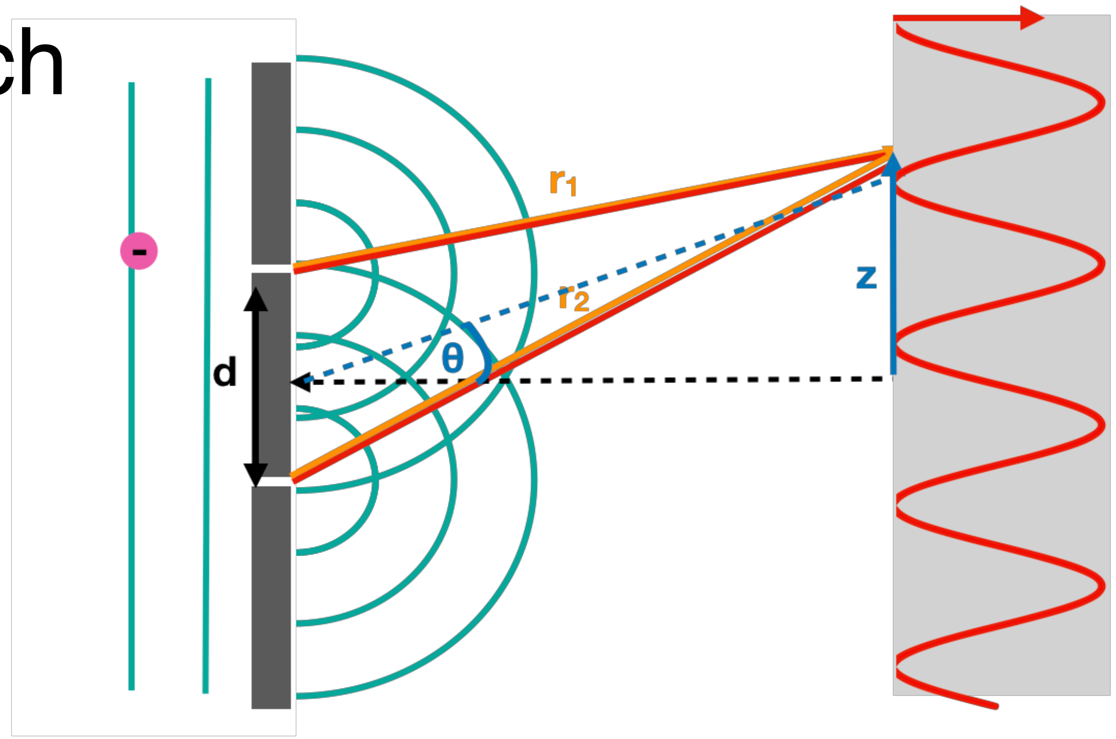


Evidence for Matter waves

- Movie shown earlier was the build-up of this electron interference pattern, one electron at a time

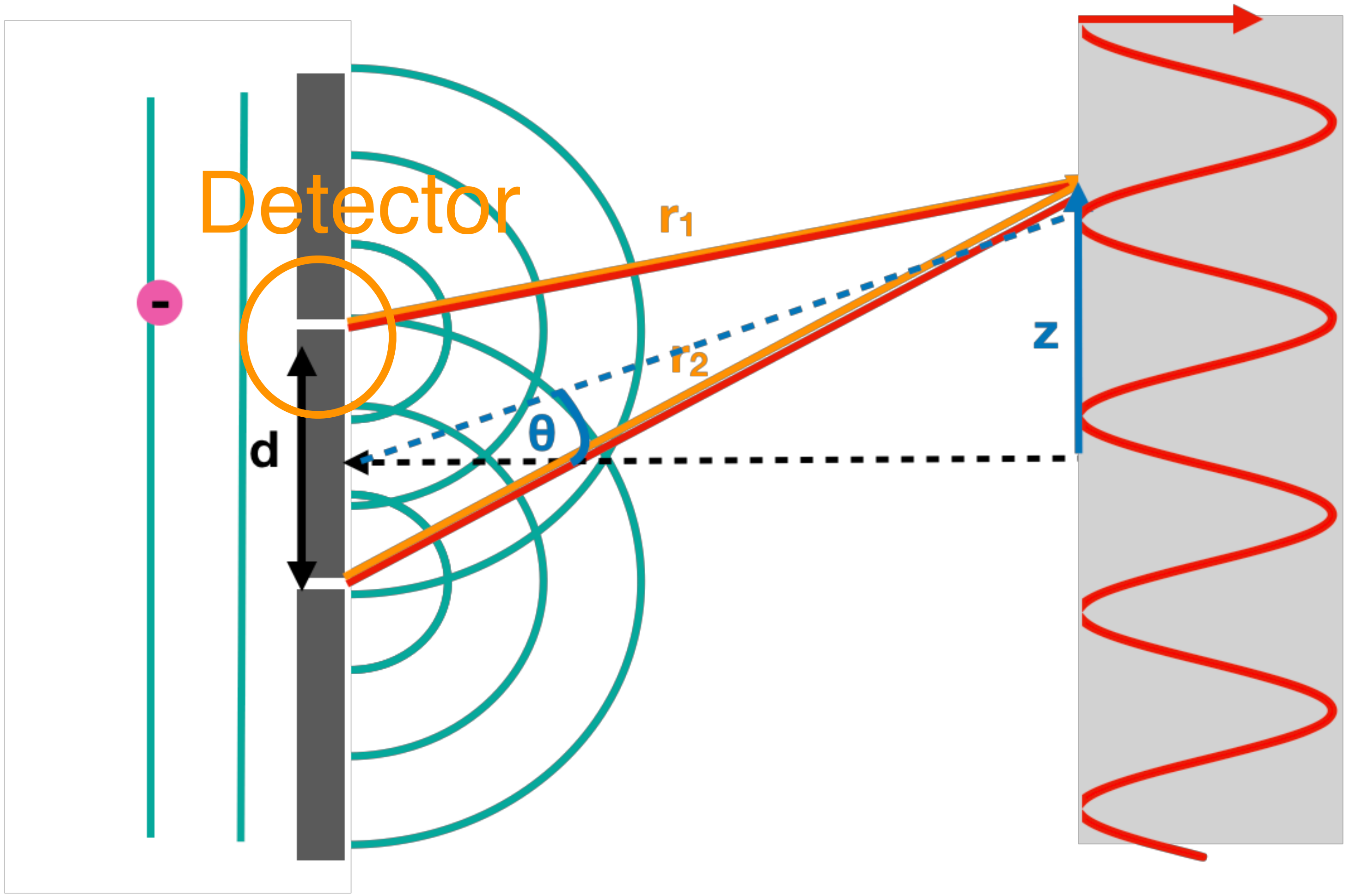
https://www.youtube.com/watch?v=hv12oB_uyFs

- This shows that each electron **interferes with itself**



Evidence for Matter waves

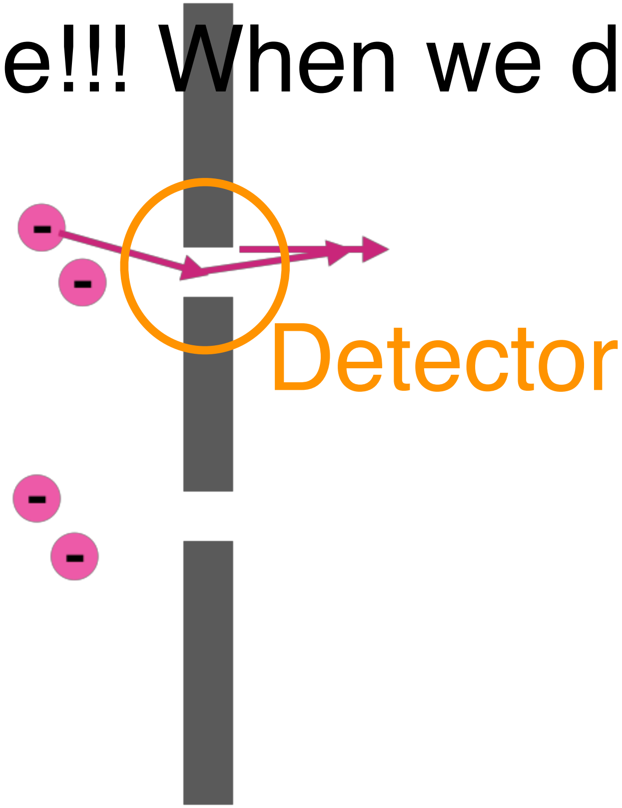
- If we in any way obtain information about **which slit the electron went through**, interference **disappears**.

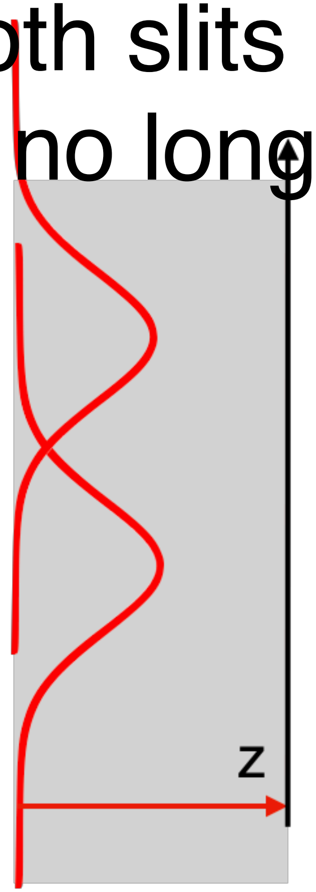


Evidence for Matter waves

- If we in any way obtain information about **which slit the electron went through**, interference **disappears**.

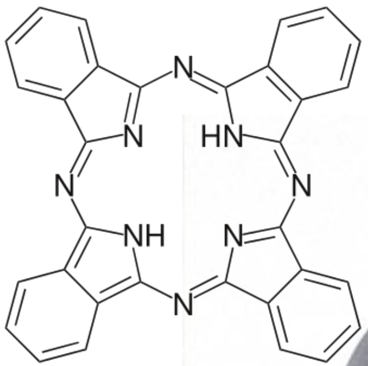
- This implies, interference only happens if the electron went through both slits at once!!! When we detect it, it no longer does



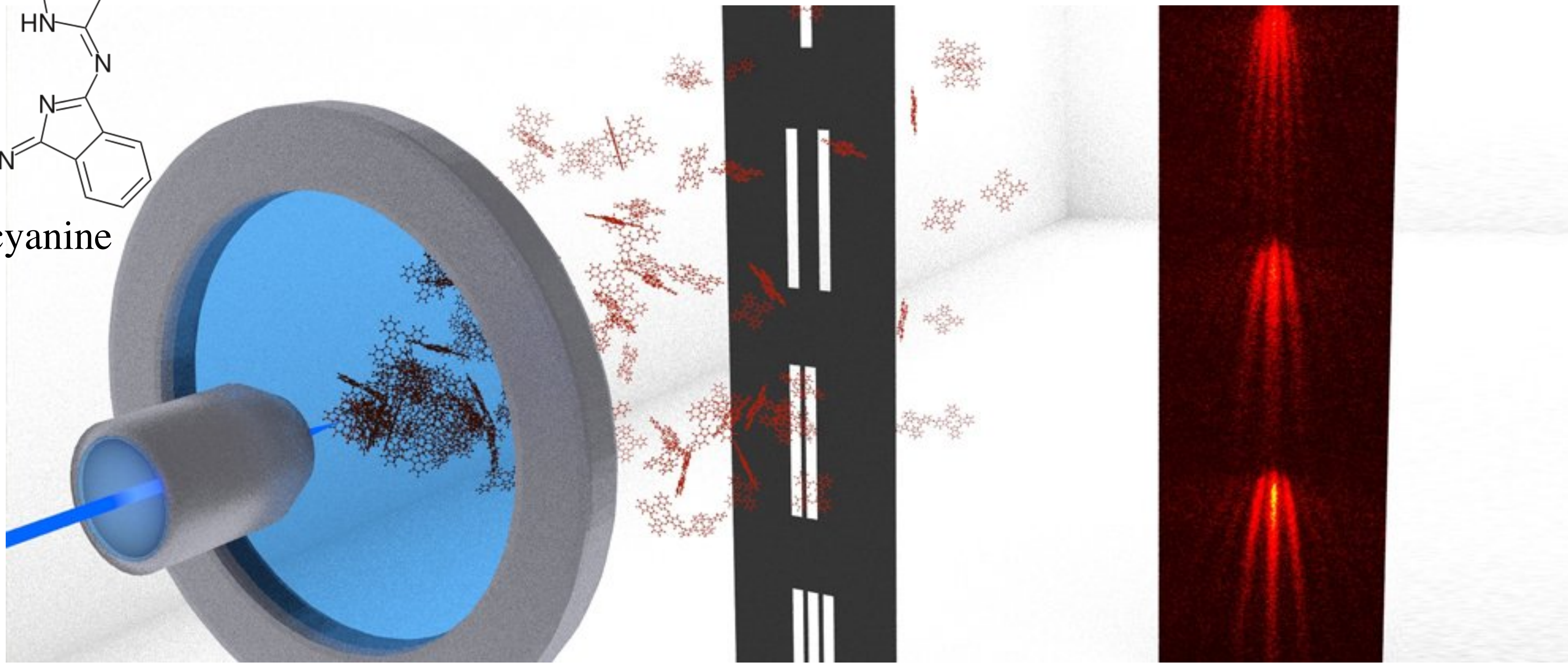


Evidence for Matter waves

By now, double slit interferencea also for neutrons, atoms, **molecules**:

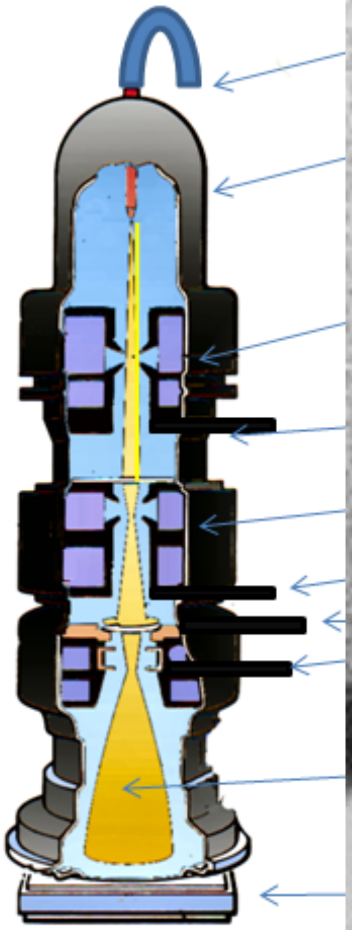


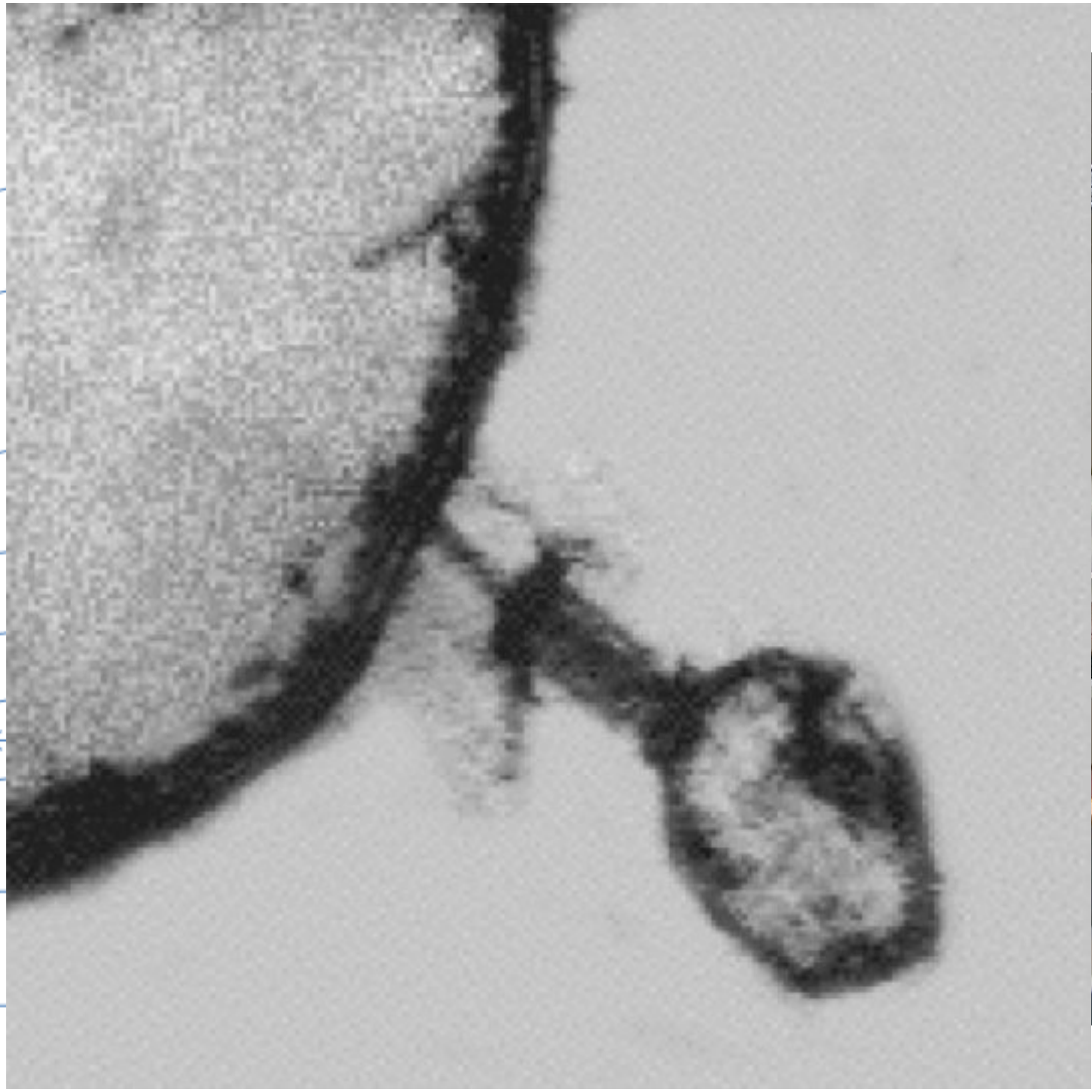
Phthalocyanine



https://www.sciencealert.com/physicists-run-a-classic-quantum-experiment-showing-how-molecules-act-as-waves

Example: Electron microscope

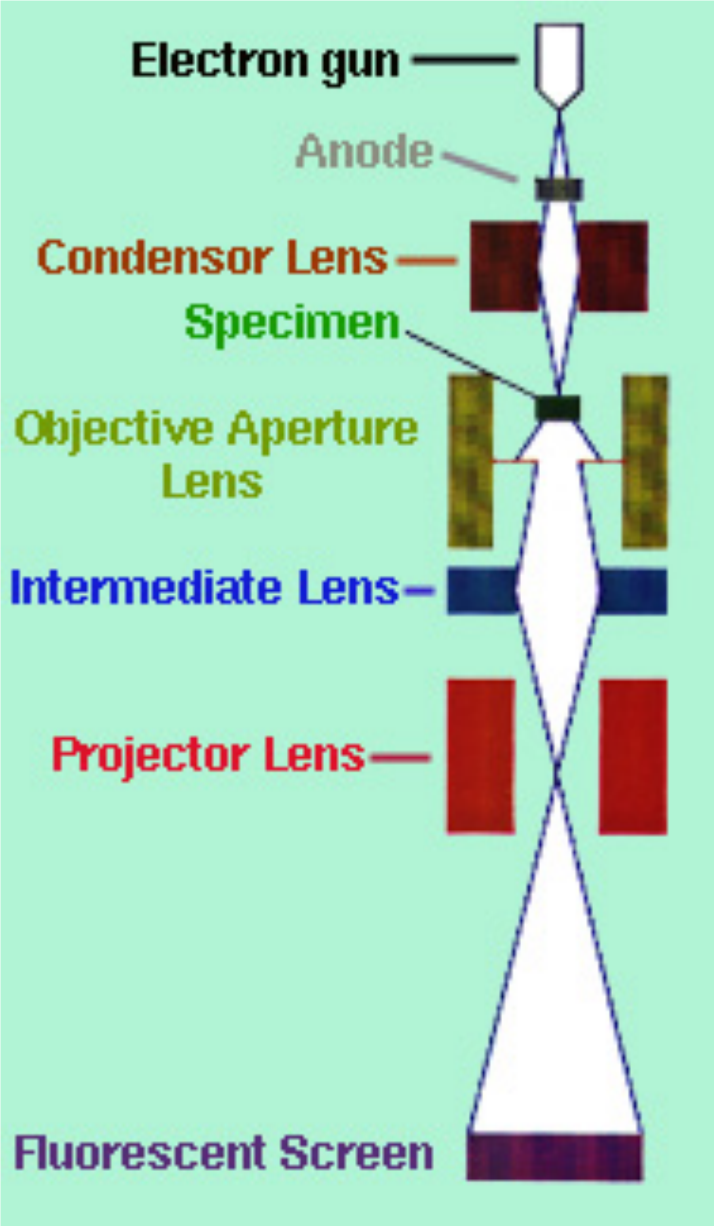


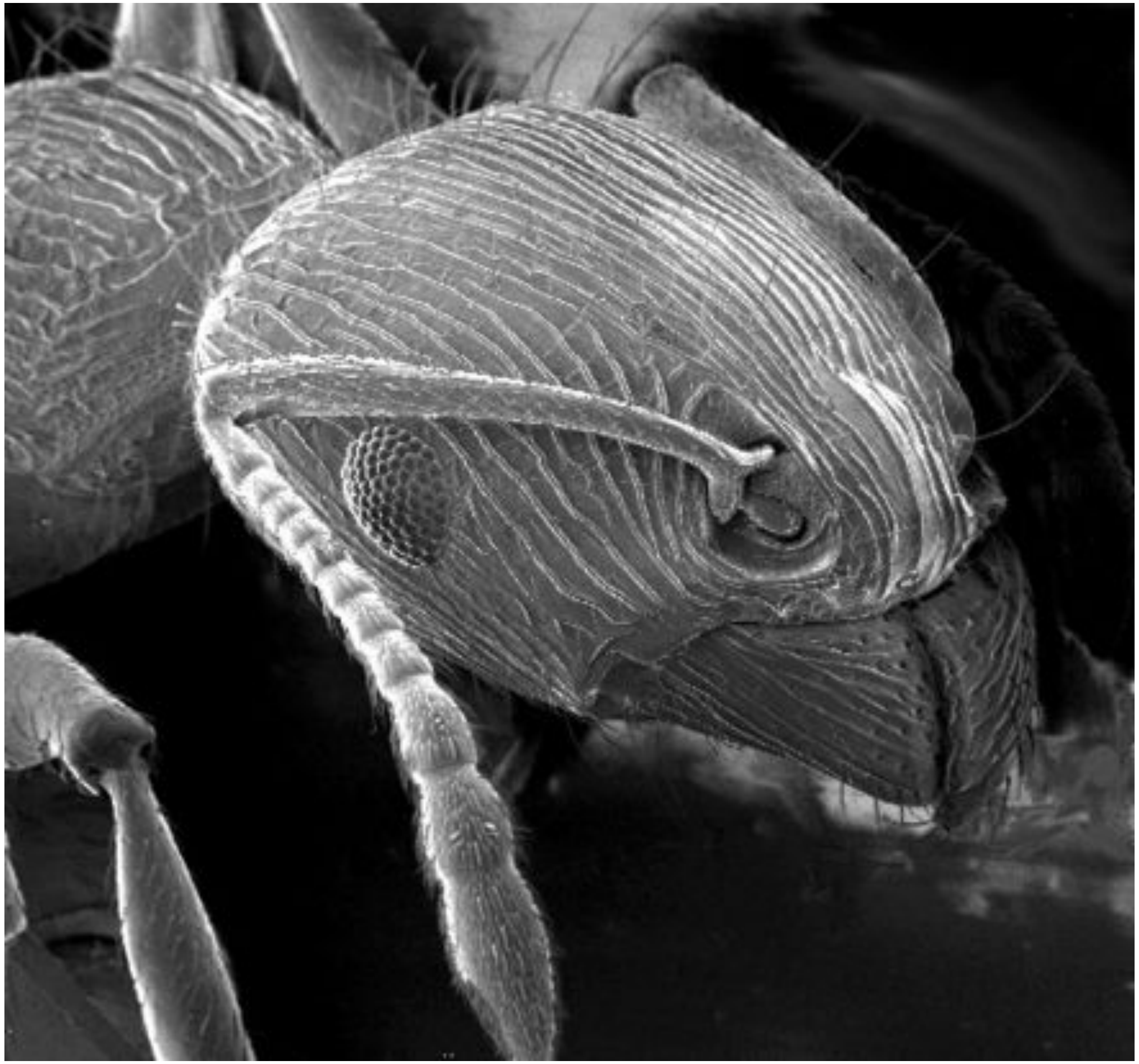


Transmission Electron Microscope

Q: Electron microscope has a much better resolution than optical one. Why?

Example: Electron microscope





Q: How do we make lenses for electron microscopes?

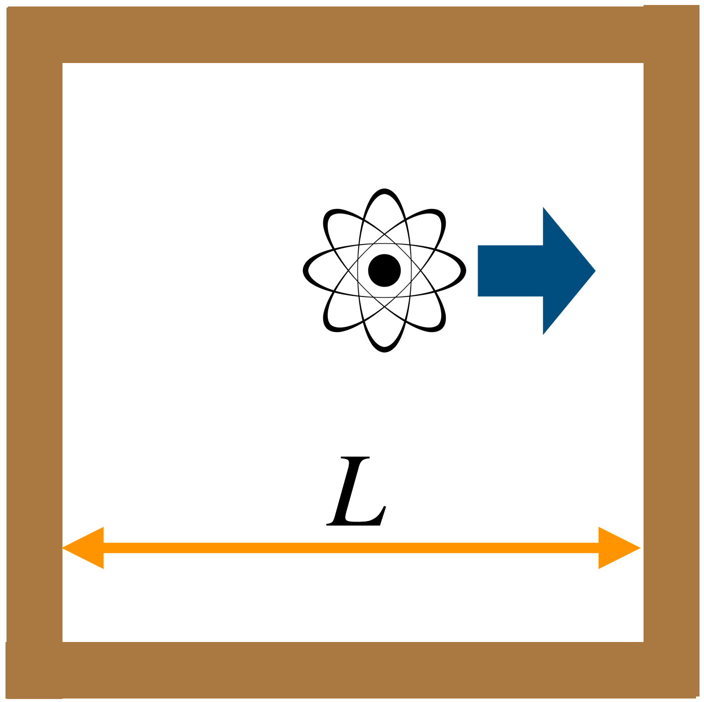
2.4.3) Particle in a box

So far: quantum theory (i) light **quanta**
(ii) Particles are also waves.

The latter leads often to (iii) **quantisation** of physical variables that were earlier continuous.

Consider particle in a box:

Q: What does this remind you of?



Particle in a box

A: Standing waves in cavity (2.1.3)

Turns out matter wave Ψ also must obey **boundary conditions** $\Psi(0) = \Psi(L) = 0$

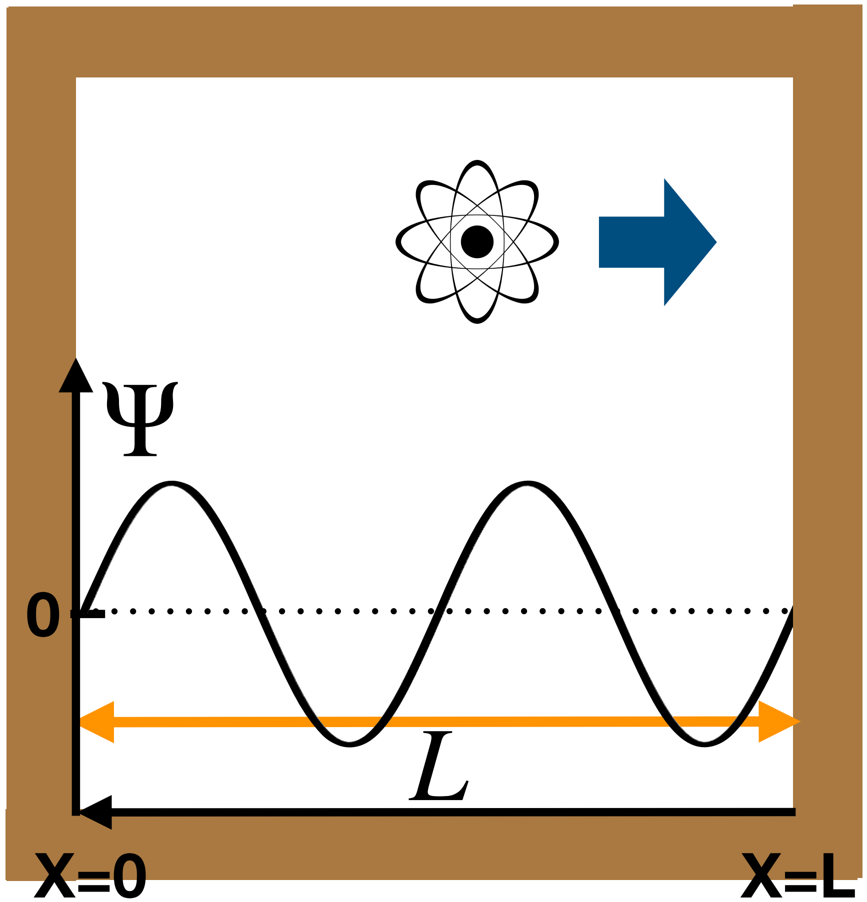
We can thus use Eq. (16)

$$\lambda_{dB} = \frac{2L}{n} \quad (61)$$

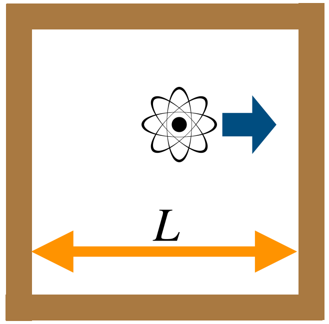
$$n = 1,2,3\ldots$$

To use Eq. (55)

$$E_{kin} = \frac{p^2}{2m} = \frac{h^2}{2m\lambda_{dB}^2} \quad (62)$$



Particle in a box



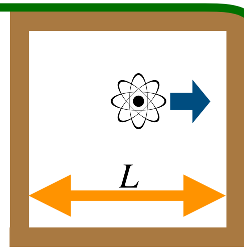
Inserting Eq. (61) into (62) we reach

Energy quantisation for a particle in a box

$$E_n = \frac{n^2 h^2}{8mL^2} \qquad n = 1,2,3\ldots \qquad (63)$$

- Particle in box cannot have any arbitrary energy
- Allowed states *n* are called **energy levels**
- The more confined (L small) the larger the energy differences between two levels
- Zero energy is **not allowed**

Examples: Particle in a box



(1) Electron $m = m_e$ in atom-size box $L = 0.1$**nm**

Eq. (63): $E_n = 38n^2$ **eV**

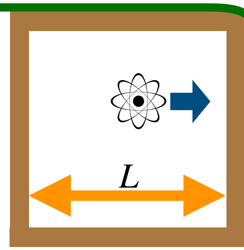
Q: Photon of 38 eV energy has $\lambda = 205$ **nm**
Spectral range?

(2) Proton $m = m_p$ in nucleus-size box $L = 4$ **fm**

Eq. (63): $E_n = 0.5n^2$ **GeV**

Q: Photon of 0.5 GeV energy….
Spectral range?

Examples: Particle in a box



(3) Marble $m = 10g$ in box $L = 10$ **cm**

Eq. (63): $E_n = 5.5 \times 10^{-64}\mathbf{J}\ n^2$

Velocity matching the *n=1* kinetic energy is

$$v = 3.3 \times 10^{-31} \textbf{ m/s}$$

We would never notice this. Quantisation irrelevant in (3), in contrast to examples (1) and (2) where it would be very important.

2.4.4) Free particles

If not in box, still want to say *where* a particle is

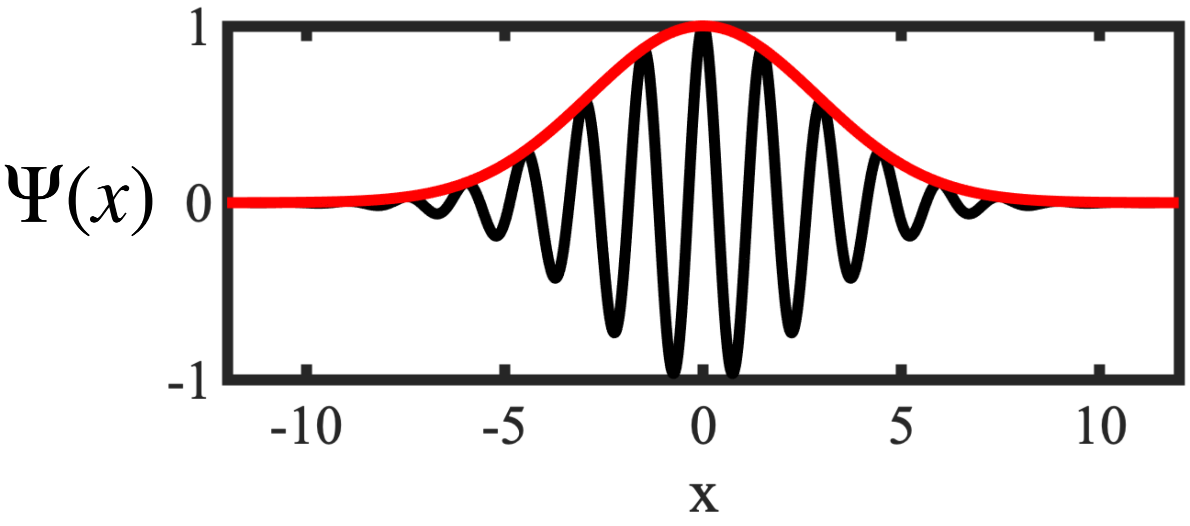
$$\Psi(x) = A\cos[2\pi(\frac{x}{\lambda_{dB}} - \nu t)]$$

Can't from Eq. (57), since *cos* is "everywhere"

Solution?

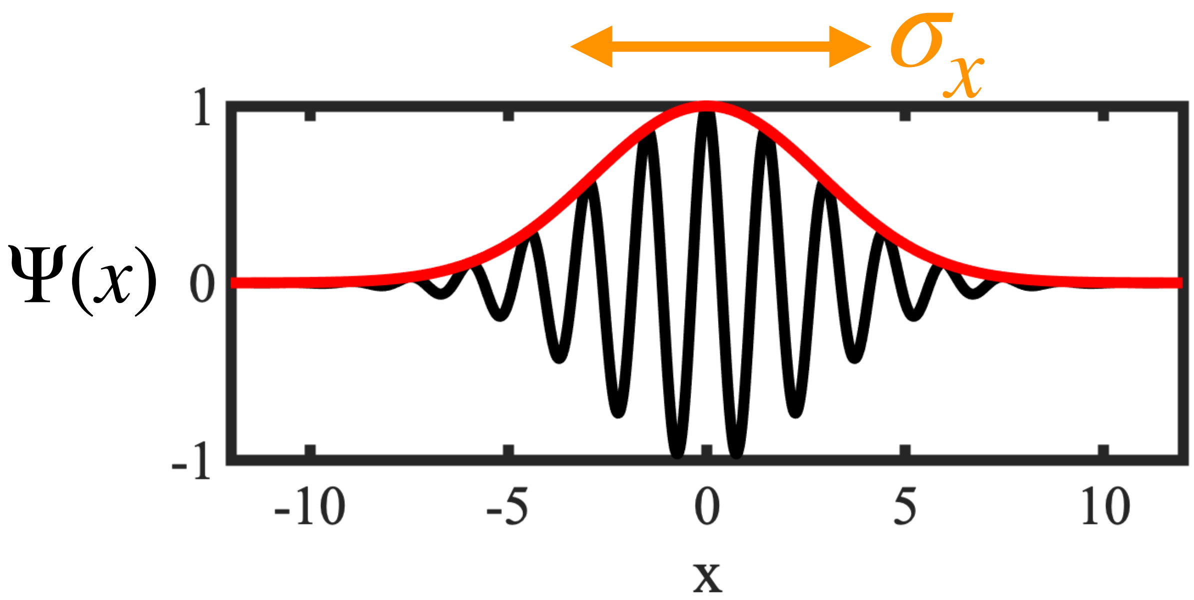
Wave-packets

(see week 5)



Free particles / Matter wave packets

Wave-packets
(see week 5)



Gaussian wavepacket
(red line)

$$\Psi(x) = \frac{1}{\sqrt{2\pi\sigma_x}} e^{-\frac{x^2}{2\sigma_x^2}}$$

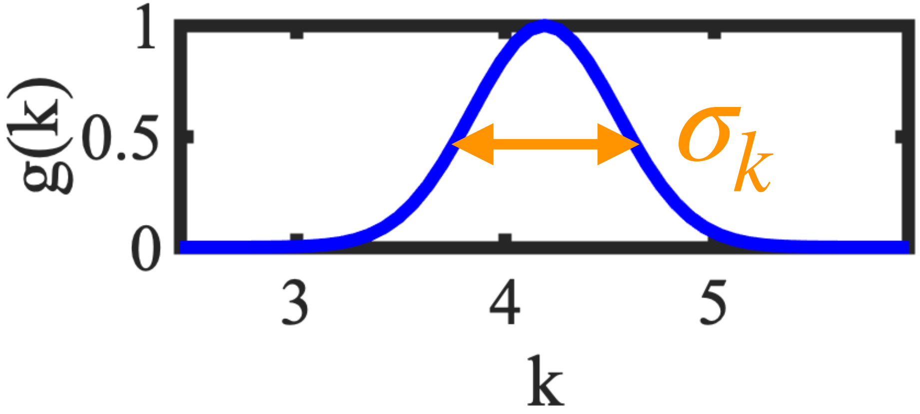
Can turn this into precise position by $\sigma_x \rightarrow 0$

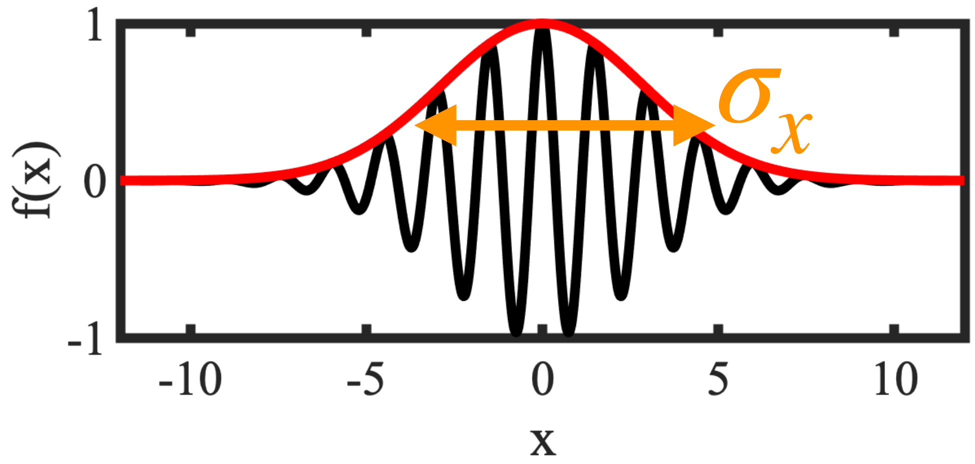
…or can we?

2.4.5) Uncertainty principle

Recall Eq. (52): x and k **widths** are **inverse**

$$\sigma_x = 1/\sigma_k$$





- But the more wave numbers k (or wavelengths λ) are part of the wave-packet, the more different **momenta** does it have!! $p = \hbar k$

Uncertainty principle

This leads is to the

Heisenberg uncertainty principle

It is impossible to know **both** the exact position **and** exact momentum of a particle/object at the same time. (64)

Uncertainty principle

This leads is to the

Heisenberg uncertainty principle

It is impossible to know **both** the exact position **and** exact momentum of a particle/object at the same time. (64)

- This is a fundamental consequence of the wave-particle duality.
- It is **not** due to measurement imperfections.
- See book for arguments based on measurement **perturbing** the particle

Uncertainty principle

Let's phrase the uncertainty principle in terms of math

For the Gaussian wavepacket we define:

Position uncertainty $$\Delta_x = \frac{1}{\sqrt{2}}\sigma_x$$

Momentum uncertainty $$\Delta_p = \frac{\hbar}{\sqrt{2}}\sigma_k$$

(65b)

Uncertainty principle

Based on our discussion of wave-packets, then

Heisenberg uncertainty principle for the Gaussian wavepacket

(quantitative version)

$$\Delta_x \Delta_p = \frac{\hbar}{2} \qquad (65)$$

- Follows from $\sigma_x = 1/\sigma_k$

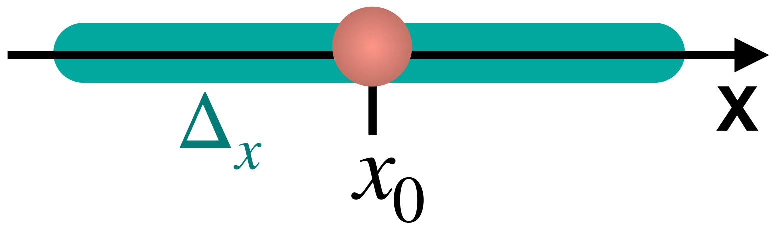
Uncertainty principle

One can show in general, that:

Heisenberg uncertainty principle for any other wavepacket
(quantitative version) $$\Delta_x \Delta_p > \frac{\hbar}{2}$$ **(65b)**

Example: Uncertainty relation

Suppose we measured proton position x_0 up to $\Delta_x = 1 \times 10^{-11}$ **m**



What do we know about position t=1 s later?

From Eq. (65) $\Delta_p = \dfrac{\hbar}{2\Delta_x}$

In 1 second:

$d = 3.15 \times 10^3$ **m**

Hence **don't** know v better than

$$\Delta v = \frac{\Delta_p}{m_p} = \frac{\hbar}{2m_p\Delta_x}$$

Might travel

$$d = \Delta_v t = = \frac{\hbar t}{2m_p\Delta_x}$$

The better we know x initially, the less well at the end

Uncertainty principle

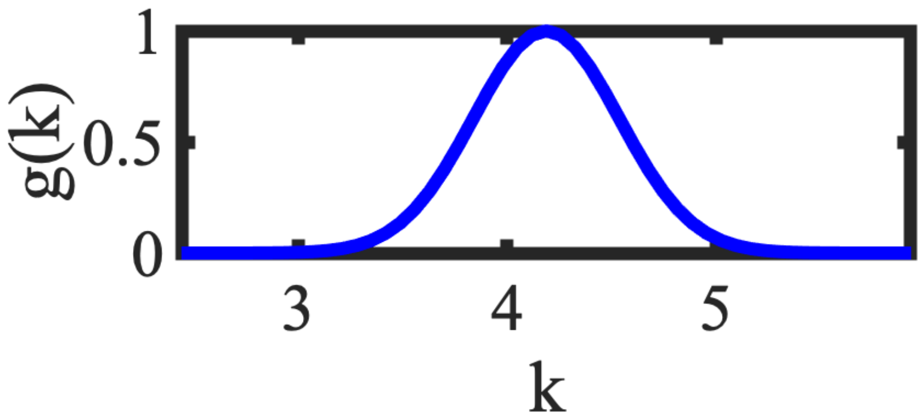
In week 5, we looked at wave packets in **space** and **wavenumber.**

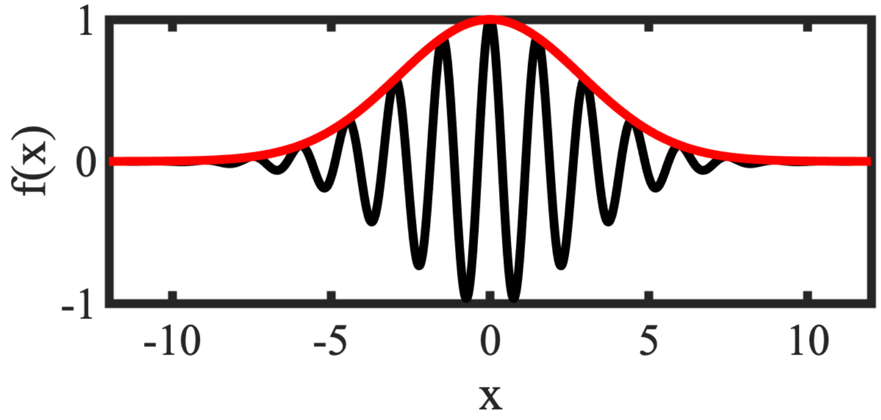
The precise same relations exist between **time** and **frequency**

Let's look briefly at **temporal wave packets**

Temporal wavepacket

Earlier spatial wavepacket $\sigma_x = 1/(2\sigma_k)$





$$f(x) \sim \int_{-\infty}^{\infty} dk \; e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \cos(kx)$$

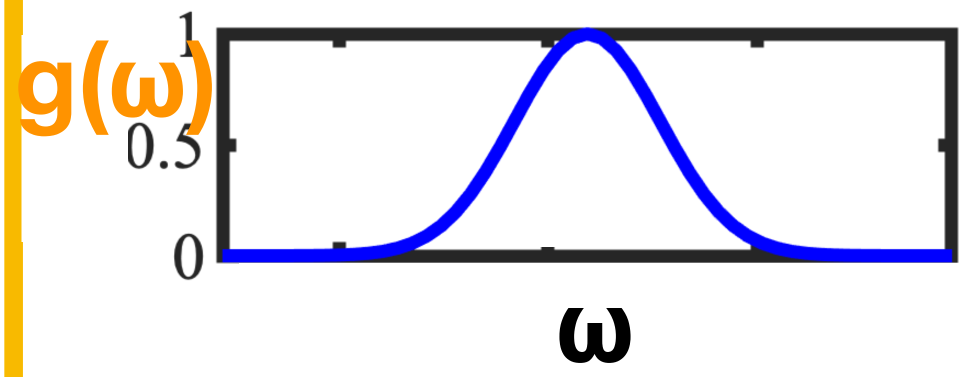
$x \to t, \; k \to \omega$

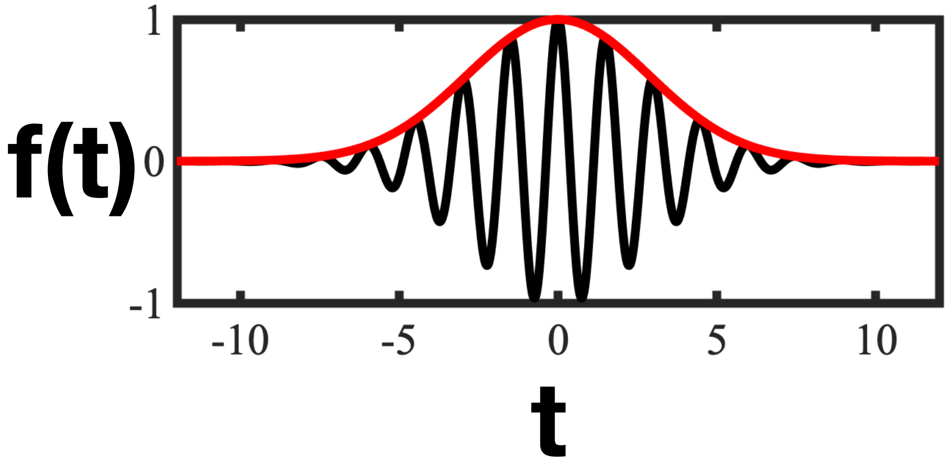
$$f(t) \sim \int_{-\infty}^{\infty} d\omega \; e^{-\frac{(\omega-\omega_0)^2}{2\sigma_\omega^2}} \cos(\omega t)$$

Temporal wavepacket

Temporal wavepacket

$$\sigma_t = 1/(2\sigma_\omega)$$





$$f(t) \sim \int_{-\infty}^{\infty} d\omega \; \underbrace{e^{-\frac{(\omega-\omega_0)^2}{2\sigma_\omega^2}}}_{g(\omega)} \cos(\omega t)$$

Uncertainty principle

Thus for the same reasons as with position and momentum we find a

Energy - time uncertainty relation

$$\Delta_E \Delta_t > \frac{\hbar}{2} \quad (66)$$

- Here Δ_E is the **energy uncertainty of some** state or process
- Δ_t is the **characteristic duration** (e.g. lifetime) associated with it

2.4.6) Matter wave velocities

We looked at E,p,x of matter-waves, but not v.

Let's try for: $\Psi(x) = A\cos[2\pi\,(\frac{x}{\lambda_{dB}} - \nu t)]$

From Eq. (8), phase velocity:

Insert Eq. (55): $\lambda_{dB} = \frac{h}{\gamma m v}$ and (56): $\nu = \frac{E}{h}$

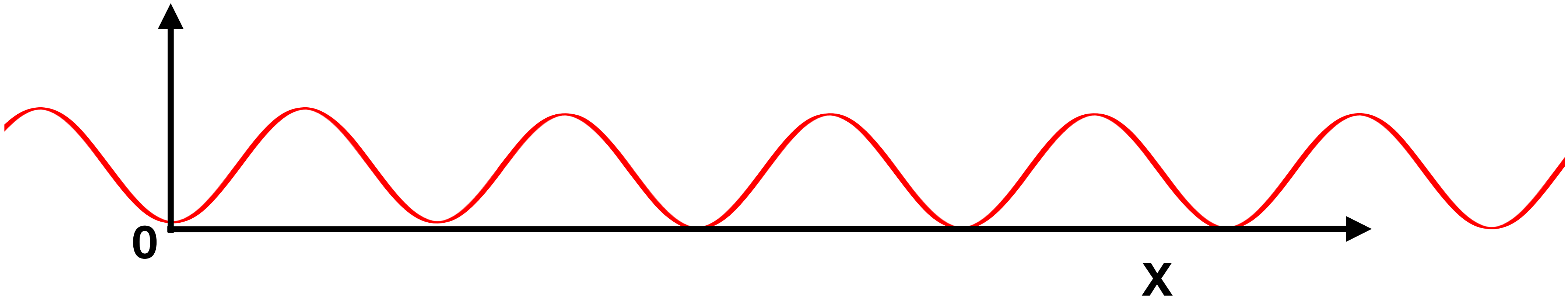
$$V = \nu\lambda_{dB} = \frac{E}{\gamma m v} = \frac{\gamma m c^2}{\gamma m v} = \frac{c^2}{v} = \frac{c}{v}c > c$$

Faster than light!! Seems irritating.….but…

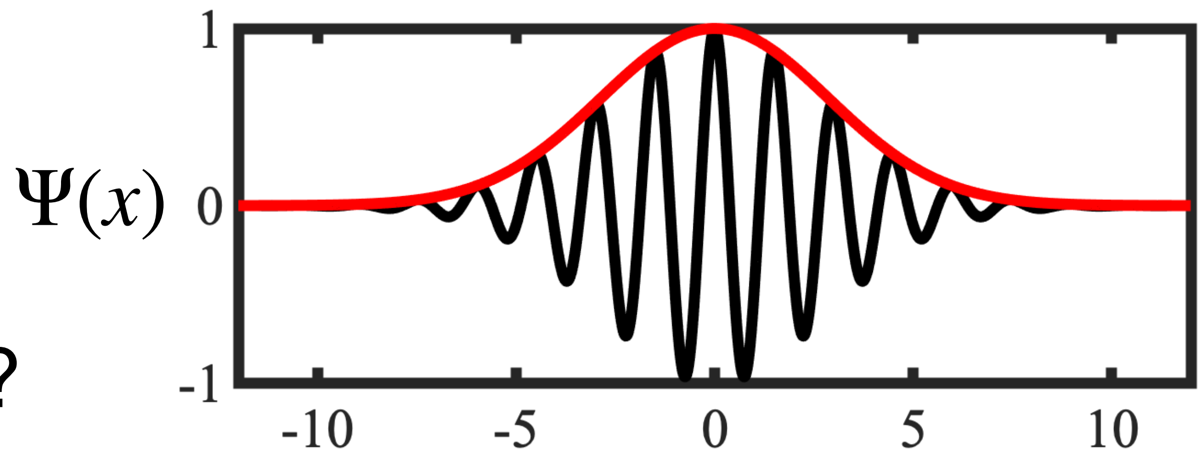
2.4.6) Matter wave velocities

But for cos matter wave $\Psi(x) = A\cos[2\pi(\frac{x}{\lambda_{dB}} - \nu t)]$

Cannot really define location either!!



For that we needed **wave-packet**



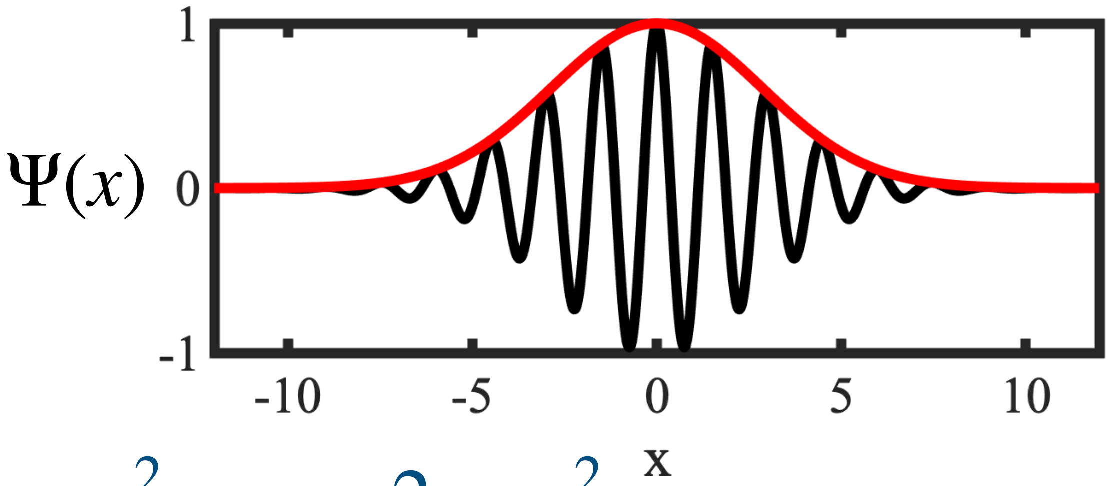
Which moves at…?

2.4.6) Matter wave velocities

Which moves at…?

Groupvelocity Eq. (54)

$$v_g = \left.\frac{d\omega}{dk}\right|_{k=k_0}$$



$$\omega = 2\pi\nu = 2\pi\frac{\gamma mc^2}{h} = \frac{2\pi mc^2}{h\sqrt{1 - V^2/c^2}}$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dV}{dk/dV}$$

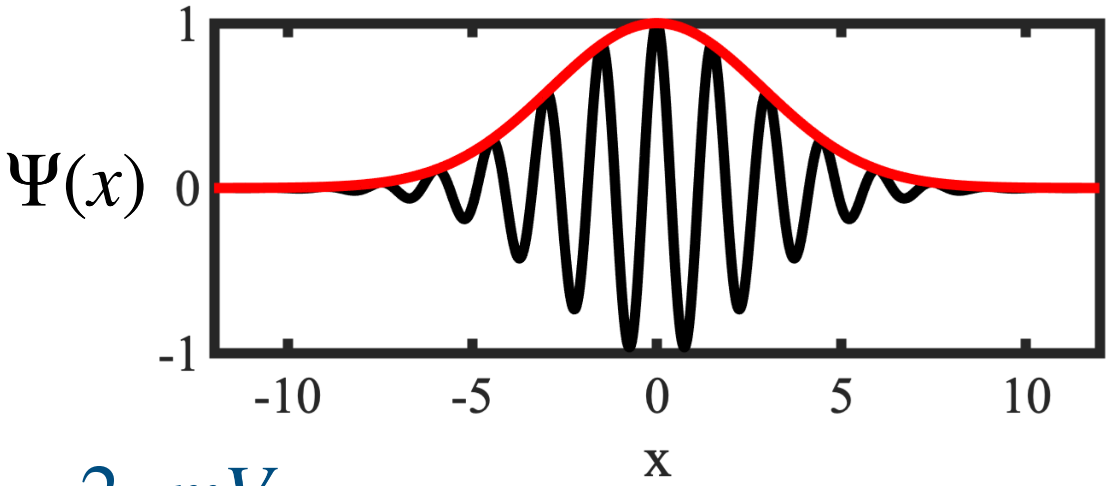
$$k = \frac{2\pi}{\lambda} = \frac{2\pi\gamma mV}{h} = \frac{2\pi mV}{h\sqrt{1 - V^2/c^2}}$$

2.4.6) Matter wave velocities

Which moves at…?

Groupvelocity Eq. (54)

$$v_g = \left.\frac{d\omega}{dk}\right|_{k=k_0}$$



$$d\omega/dV = \frac{2\pi m V}{h(1 - V^2/c^2)^{3/2}}$$

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dV}{dk/dV} = V$$

$$dk/dV = \frac{2\pi m}{h(1 - V^2/c^2)^{3/2}}$$

Particle velocity = group velocity