Week 4

PHY 106 Quantum Physics
Instructor: Sebastian Wüster, IISER Bhopal, 2018

These notes are provided for the students of the class above only.
There is no warranty for correctness, please contact me if you spot a mistake.

2.2) Particle properties of waves

Usual distinction:

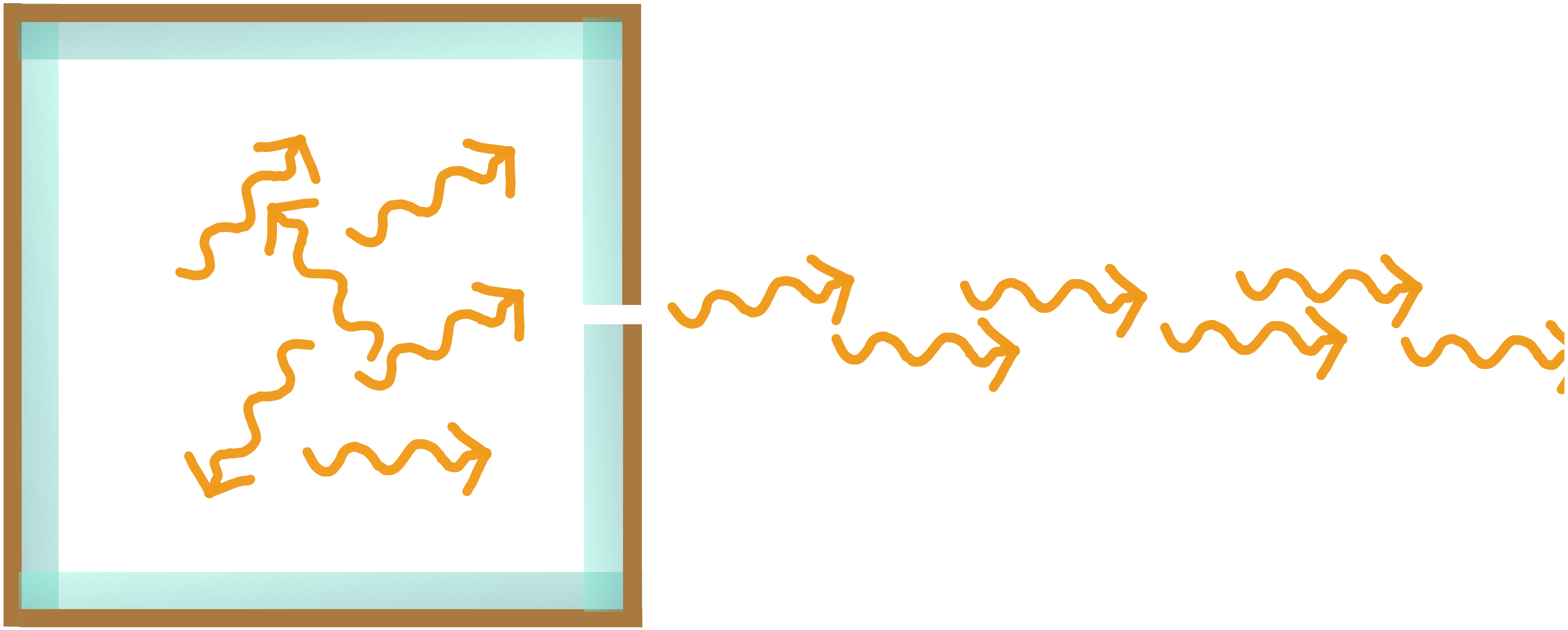
Waves: Diffraction, interference, superposition

Particle: Straight motion, moves as one

Movie: problems with photo-effect and BBR can be resolved if light is particle **and** wave

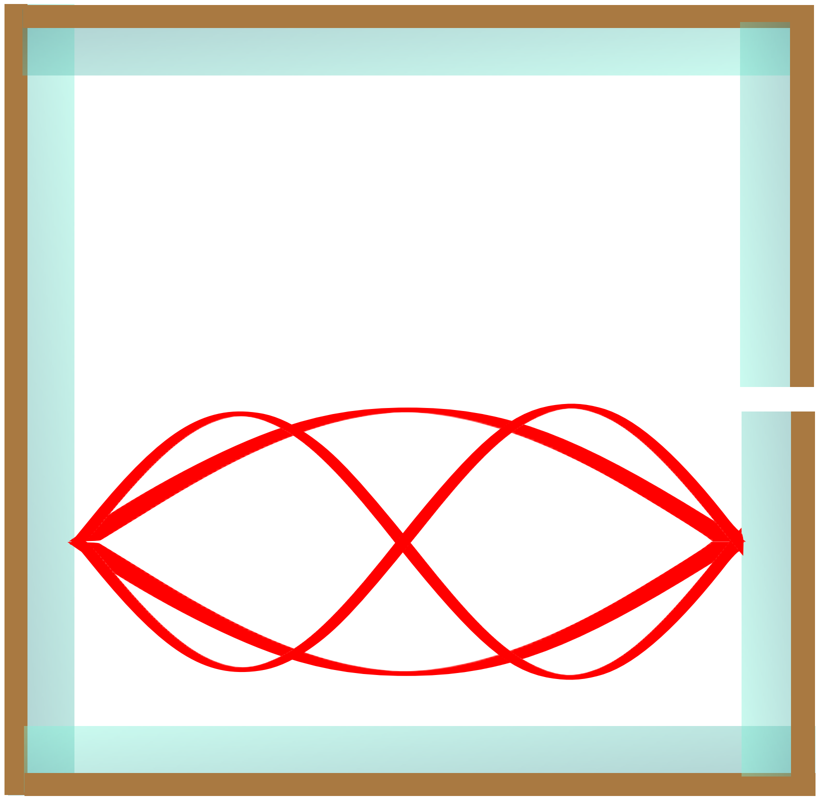
2.2.1) Black-body radiation

Now more specific black-body: mirror cavity at temperature T:



Black-body radiation

Cavity supports certain **standing waves** (see 2.1.3)



Need to consider all three dimensions

Wavelengths are fixed by Eq. 16 $L = n\frac{\lambda}{2}$

Then find frequency using Eq. 10 $\nu\lambda = c$

Black-body radiation

In this way, we find the

Density of all standing waves in a cavity

3D!!

$$G(\nu)d\nu = \frac{8\pi\nu^2}{c^3}d\nu \quad \textbf{(20)}$$

- $G(\nu)d\nu$ is the number of standing waves that have a frequency between ν **and** $\nu + d\nu$

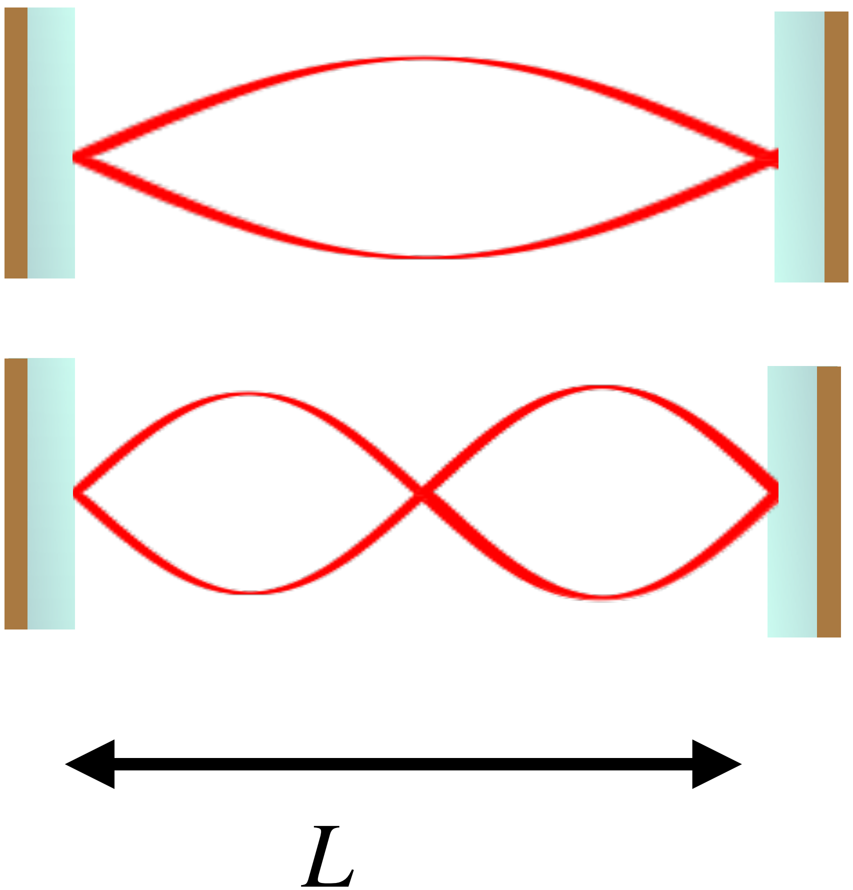
- Proof for 3D (assignment or books, not exam relevant for this course)

- To understand what type of quantity $G(\nu)d\nu$ is, let's look at the following 1D example

Example, density of waves (1D):

Eq. (16) tells us which standing waves fit into a box of size L.

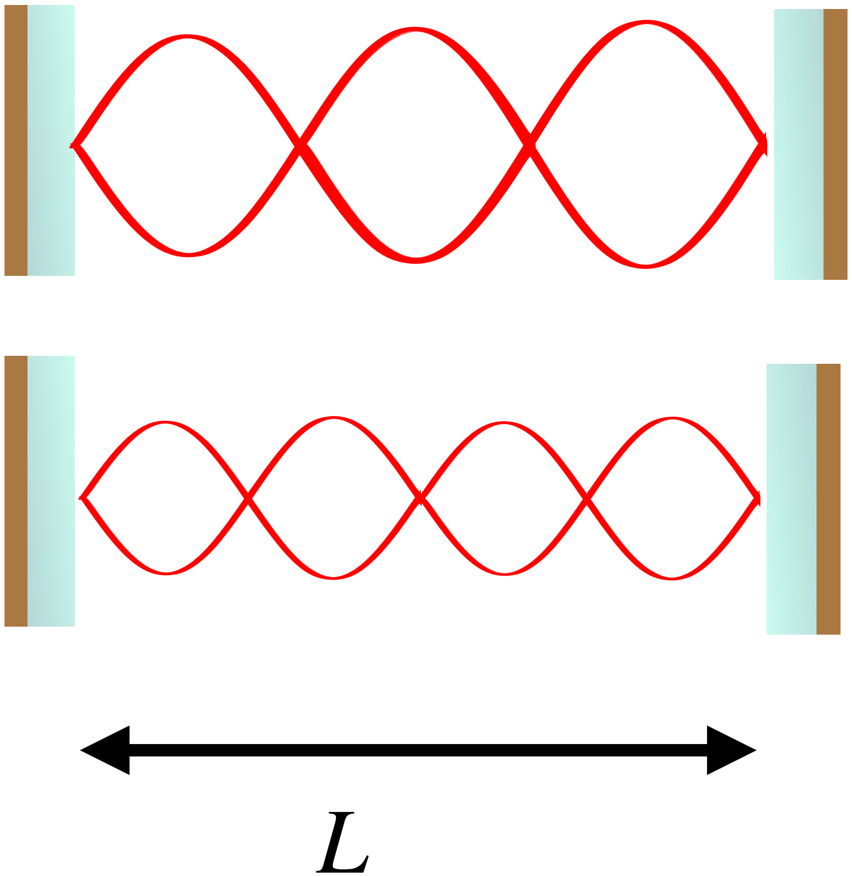
Eq. (10)



$$\lambda = 2L, \nu = \frac{c}{2L} \quad n = 1$$

$$\lambda = L, \nu = \frac{c}{L} \quad n = 2$$

Example, density of waves (1D):



$$\lambda = \frac{2L}{3},\ \nu = \frac{3c}{2L}, n = 3$$

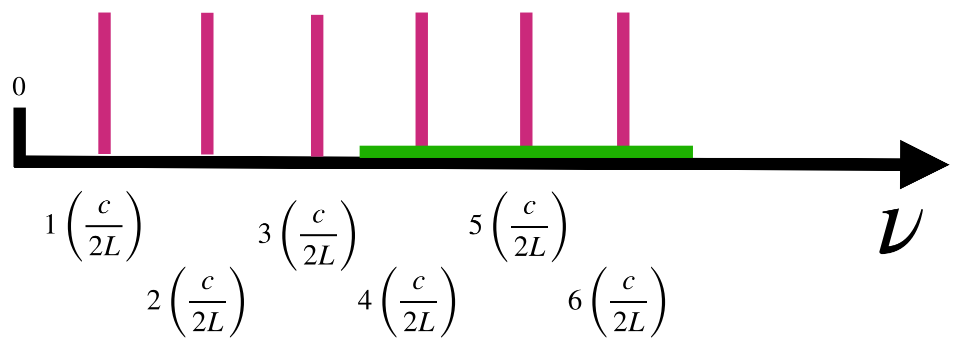
$$\lambda = \frac{L}{2},\ \nu = \frac{2c}{L}, n = 4$$

Allowed values of frequency are each bigger than the previous

$$\Delta\nu = \frac{c}{2L}$$

Example, density of waves (1D):

Let's make a diagram of allowed frequencies:



Suppose the green bar is a frequency interval of length 1

Then from diagram, the number of allowed frequencies in this interval

$$G(\nu)d\nu = \frac{1}{\left(\frac{c}{2L}\right)} = \frac{2L}{c}$$

This is the **density of standing waves in a 1D cavity**. Eq. (20) is the 3D version of it.

Black-body radiation

We assume all the electro-magnetic waves in the cavity are in **thermal equilibrium** with the walls/mirrors at temperature T.

Thermal energy per wave

$$E_{therm} = k_B T \qquad (21)$$

Boltzmann's constant

$$k_B = 1.381 \times 10^{-23} J/K \qquad (22)$$

Black-body radiation

Energy between frequencies ν **and** $\nu + d\nu$

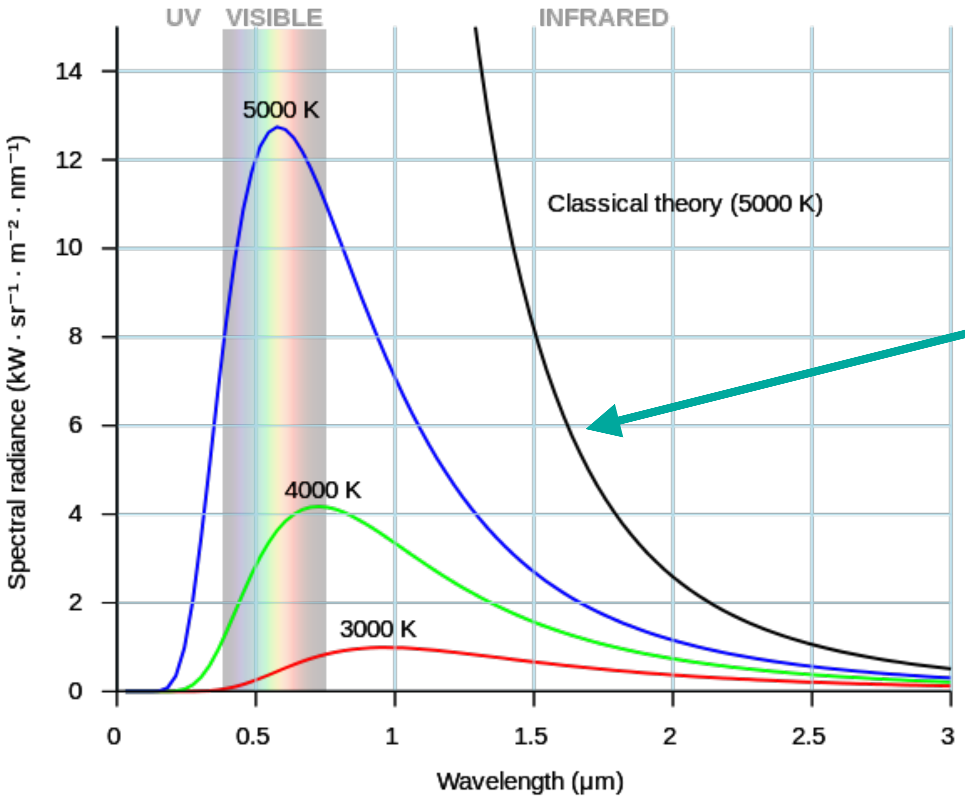
Energy per wave x No of waves

$$u(\nu)d\nu = E_{therm} G(\nu)d\nu = \frac{8\pi k_B T}{c^3}\nu^2 d\nu \quad (22)$$

Rayleigh Jeans radiation formula

- $u(\nu)\nu$ is the total energy per unit volume in the cavity within waves of frequency between ν **and** $\nu + d\nu$

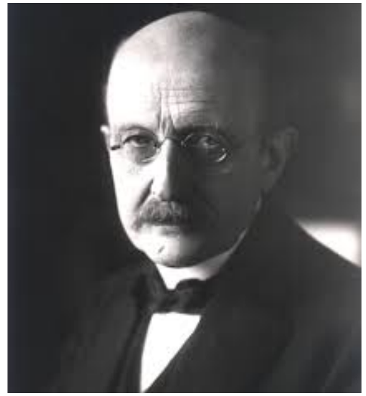
Black-body radiation



$$u(\nu)d\nu = \frac{8\pi k_B T}{c^3}\nu^2 d\nu$$

Ultra-violet catastrophe: Classical theory cannot be right at short wavelengths. Works ok at large wavelengths.

Black-body radiation



Max Planck's idea:

Standing waves can only have …

Discrete energies (energy **quantisation**)

$$E_n = nh\nu \qquad n = 0,1,2,\ldots \tag{23}$$

Planck's constant

$$h = 6.626 \times 10^{-34}\, J\, s \tag{24}$$

Black-body radiation

Quantisation changes the mean thermal energy (earlier Eq. 21):

Thermal energy per (bosonic) quantum mechanical oscillator (here standing wave)

$$E_{therm} = \frac{h\nu}{e^{h\nu/(k_B T)} - 1} \qquad (25)$$

- Nothing changes at large wavelengths/ small frequencies compared to Eq. (21)

$$\frac{h\nu}{e^{h\nu/(k_B T)} - 1} \approx_{\nu \to 0} \frac{h\nu}{(1 + (h\nu)/(k_B T) + \ldots) - 1} = k_B T$$

Bonus: Planck distribution law

Assume energy can only be $E_n = nh\nu$

From statistical physics this has probability (Boltzmann law)

$$p(n) = \frac{\exp[-E_n/(k_B T)]}{\sum_{n=0}^{\infty} \exp[-E_n/(k_B T)]} \quad \textbf{(26)}$$

Mean energy is thus:

$$E_{therm} = \sum_n E_n p(n) = \frac{\sum_{n=0}^{\infty} (nh\nu) \exp[-E_n/(k_B T)]}{\sum_{n=0}^{\infty} \exp[-E_n/(k_B T)]}$$

Bonus: Planck distribution law

Using math tricks for evaluation of series, or *Mathematica*:

```
Sum[nn h nu Exp[-nn h nu/(kB T)], {nn, 0, \[Infinity]}]/
Sum[ Exp[-nn h nu/(kB T)], {nn, 0, \[Infinity]}]
```

We finally find

$$\frac{\sum_{n=0}^{\infty}(nh\nu)\exp[-E_n/(k_B T)]}{\sum_{n=0}^{\infty}\exp[-E_n/(k_B T)]} = \frac{h\nu}{e^{h\nu/(k_B T)}-1}$$

which is Eq. (25)

$$E_n = nh\nu$$

Black-body radiation

Using Eq. (25) instead of Eq. (21), we find ….

Energy per wave x No of waves

$$u(\nu)d\nu = E_{therm}G(\nu)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3 d\nu}{e^{h\nu/(k_B T)} - 1} \quad \textbf{(27)}$$

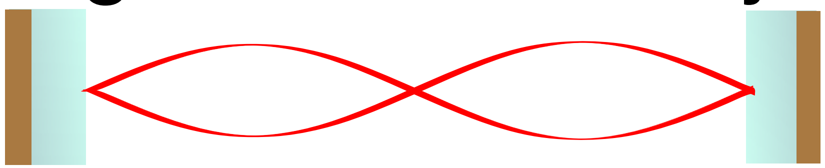
Planck's radiation formula

- Agrees with Rayleigh-Jeans for **small** freq. Stays **regular** for **large** frequencies.
- Planck's constant *h* in (24) can now be inferred by matching (26) with experiments.

Black-body radiation

- Planck did not know **why** Eq. (23) would be true
- He considered it first merely a calculation trick
- Now we know that it is due to the elm field being made of **photons**, which are particles and waves

Waves: Standing waves in cavity/blackbody!



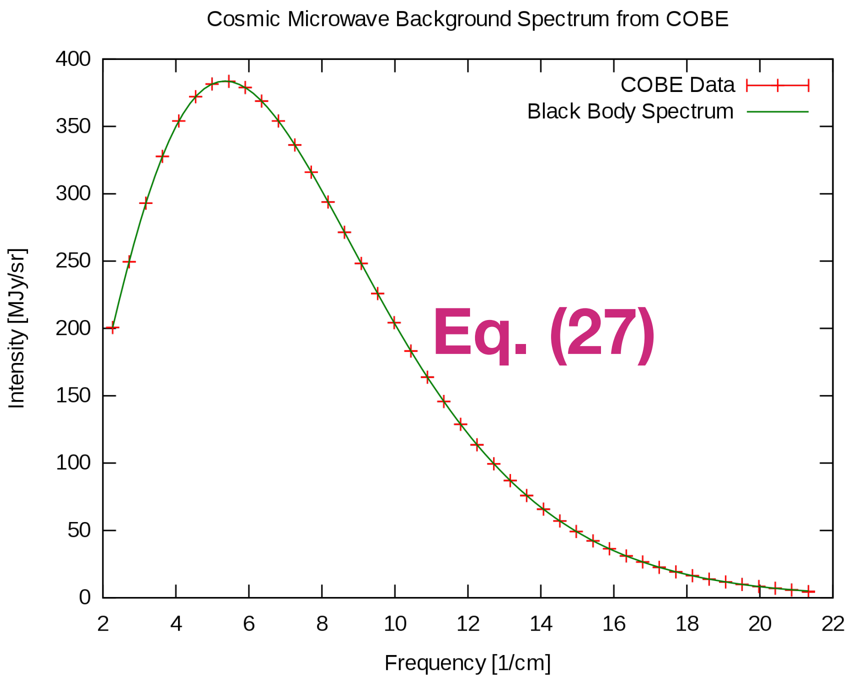
Particle: There must be an **integer** number of photons $E_n = nh\nu$

Black-body radiation

Example: Cosmic Microwave background

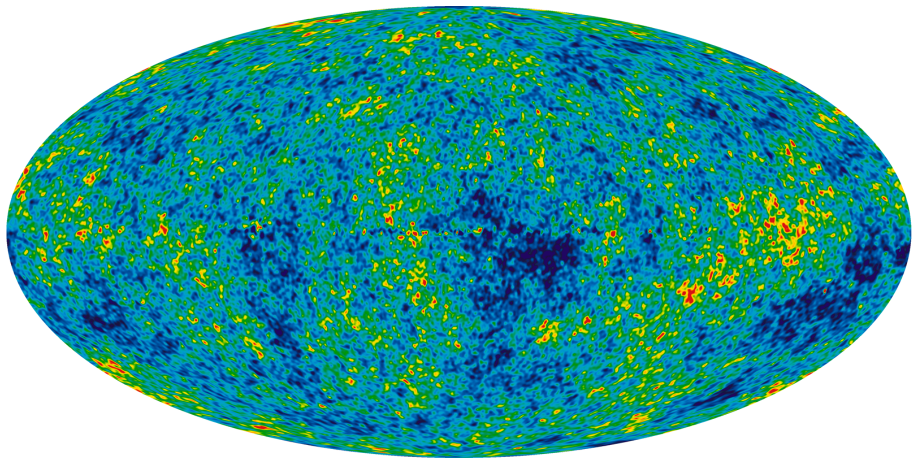
Best black body is our **universe** at T=2.72548±0.00057 K. This is remnant heat from the big bang.

Planck spectrum:



Fluctuation map:

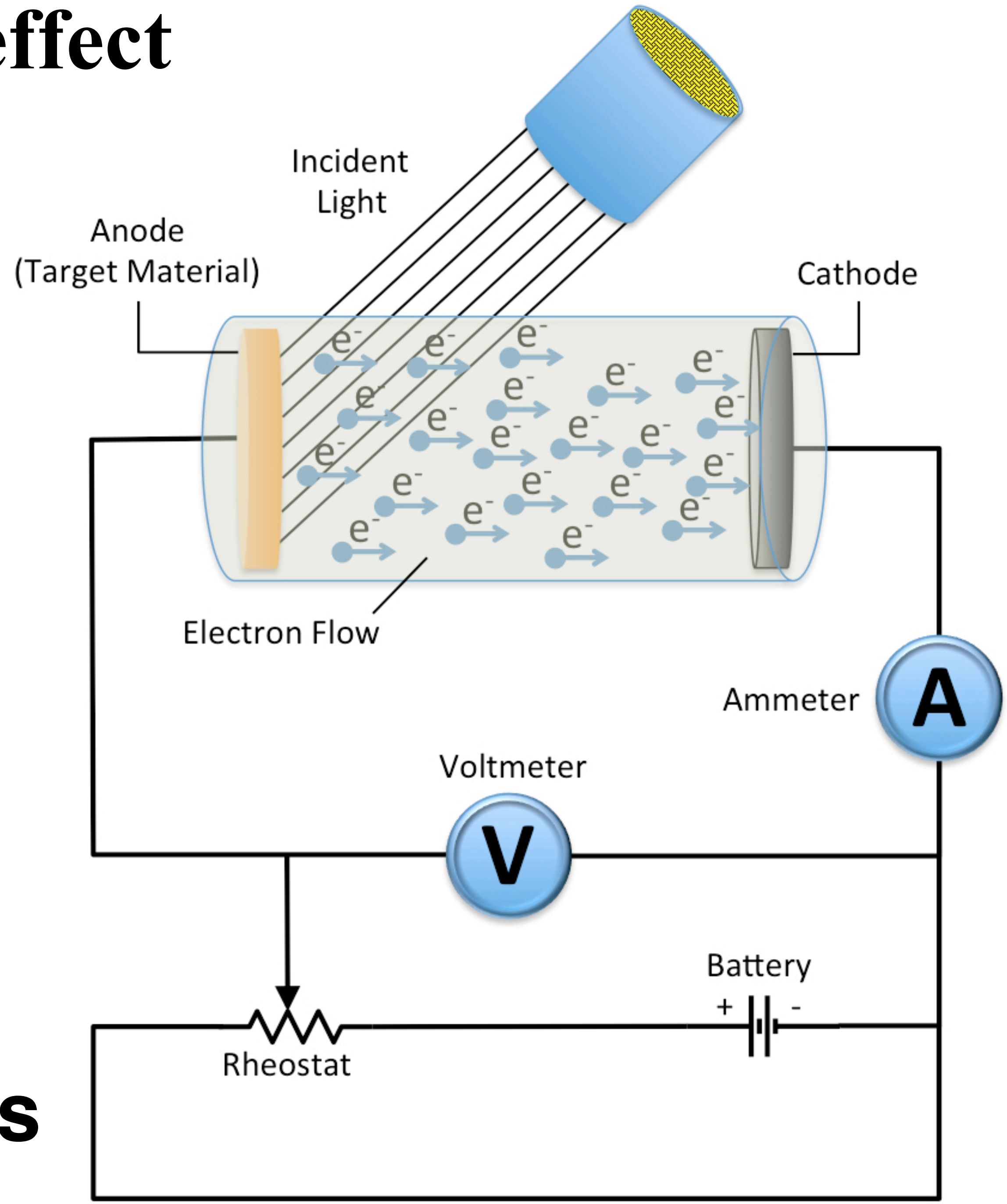
$\mathcal{O} \approx 0.0002K$



we don't understand 95% pie chart

2.2.2) Photo-electric effect

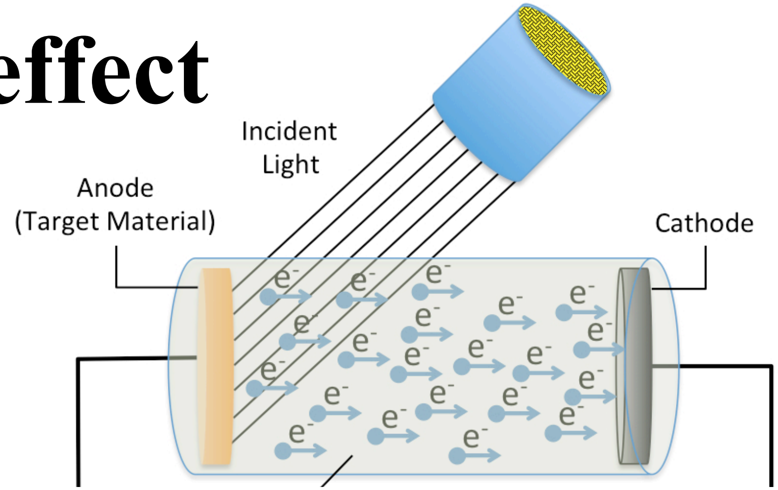
- Shine light on anode
- Cathode has a slight negative potential
- Thus electrons need a **minimal kinetic energy** to reach it
- Measure current = **number of electrons**

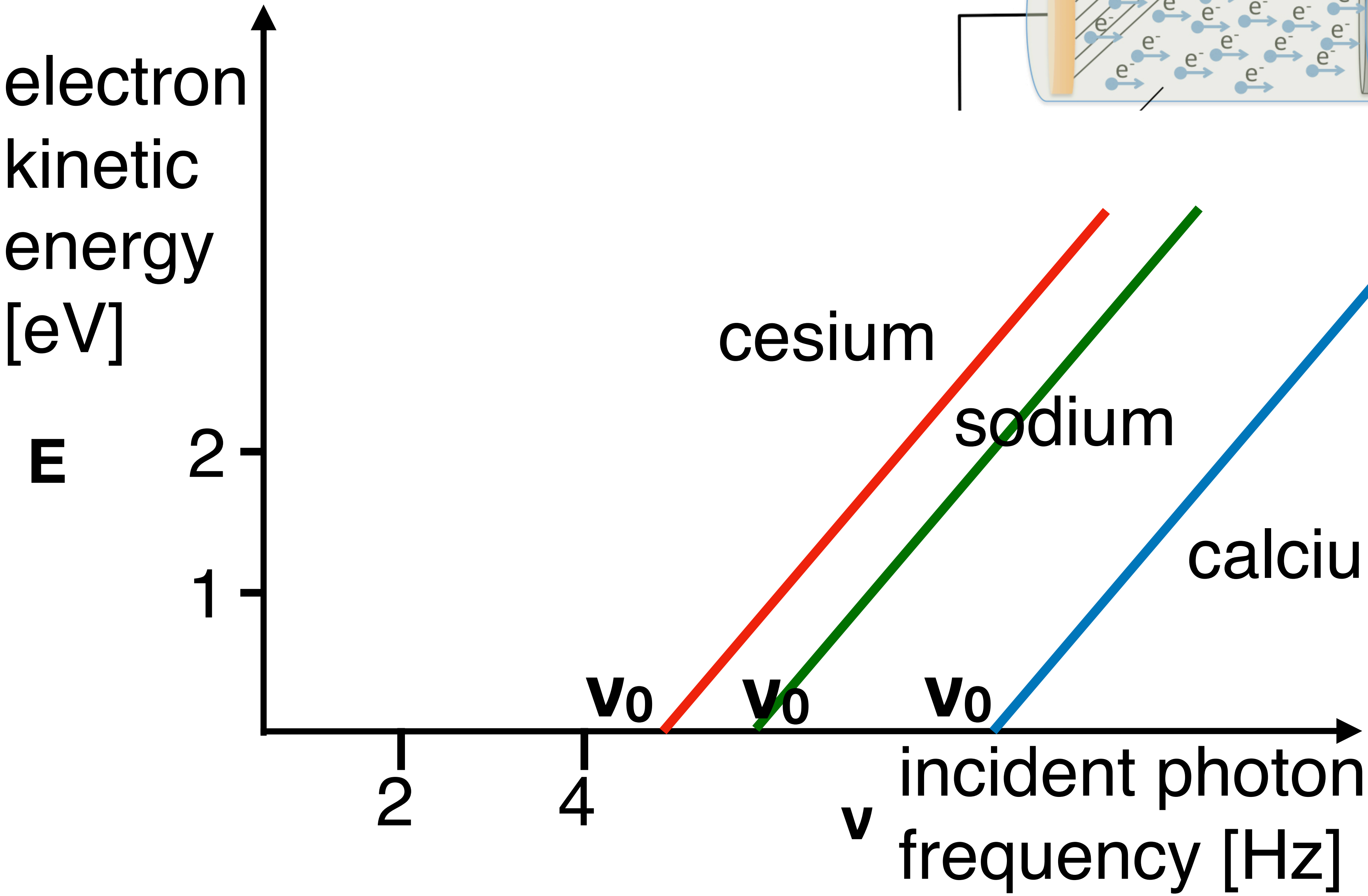


http://vlab.amrita.edu/?sub=1&brch=195&sim=840&cnt=1

Photo-electric effect

Typical measurement:





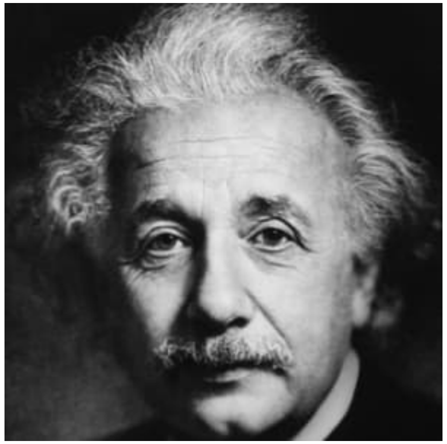
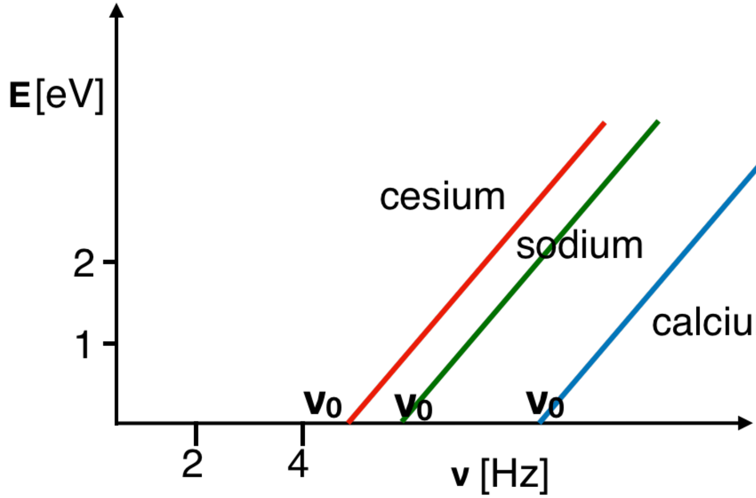


Photo-electric effect



Einstein solution

- Light comes as discrete quanta (photons)
- **photon energy:** $E_{phot} = h\nu = \hbar\omega$ (28)

(reduced Planck's constant $\hbar = \dfrac{h}{2\pi}$)

- There is at most one electron "kicked out" per photon

Photo-electric effect

then

max. **Kinetic energy of photo-electron** $$KE_{max} = h\nu - \Phi \tag{29}$$

- Φ is the work function of the material (energy to liberate electron)

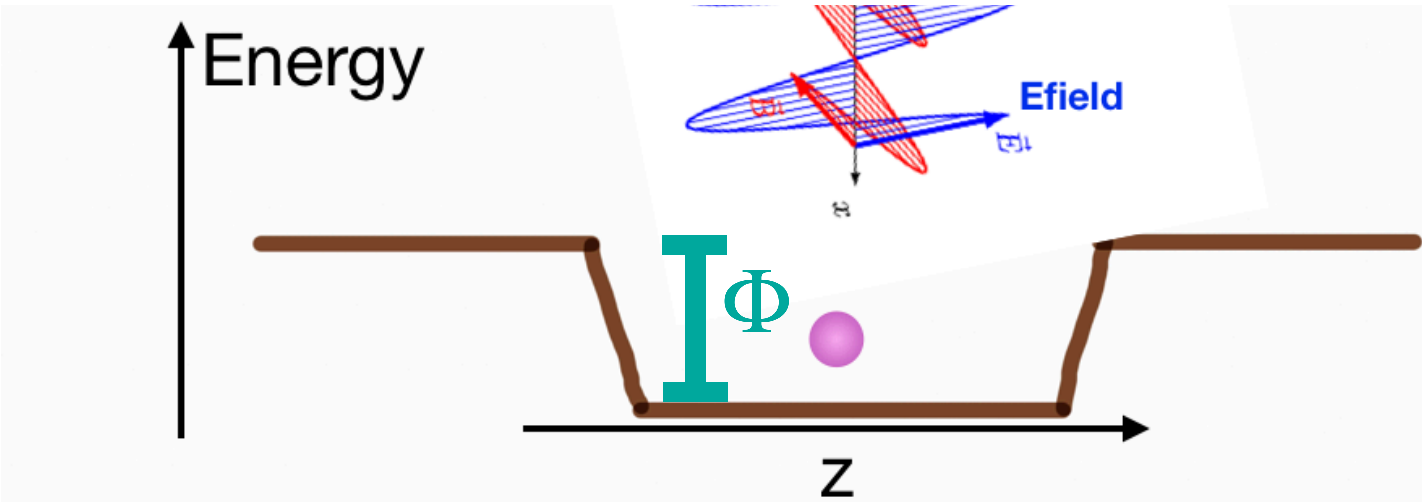
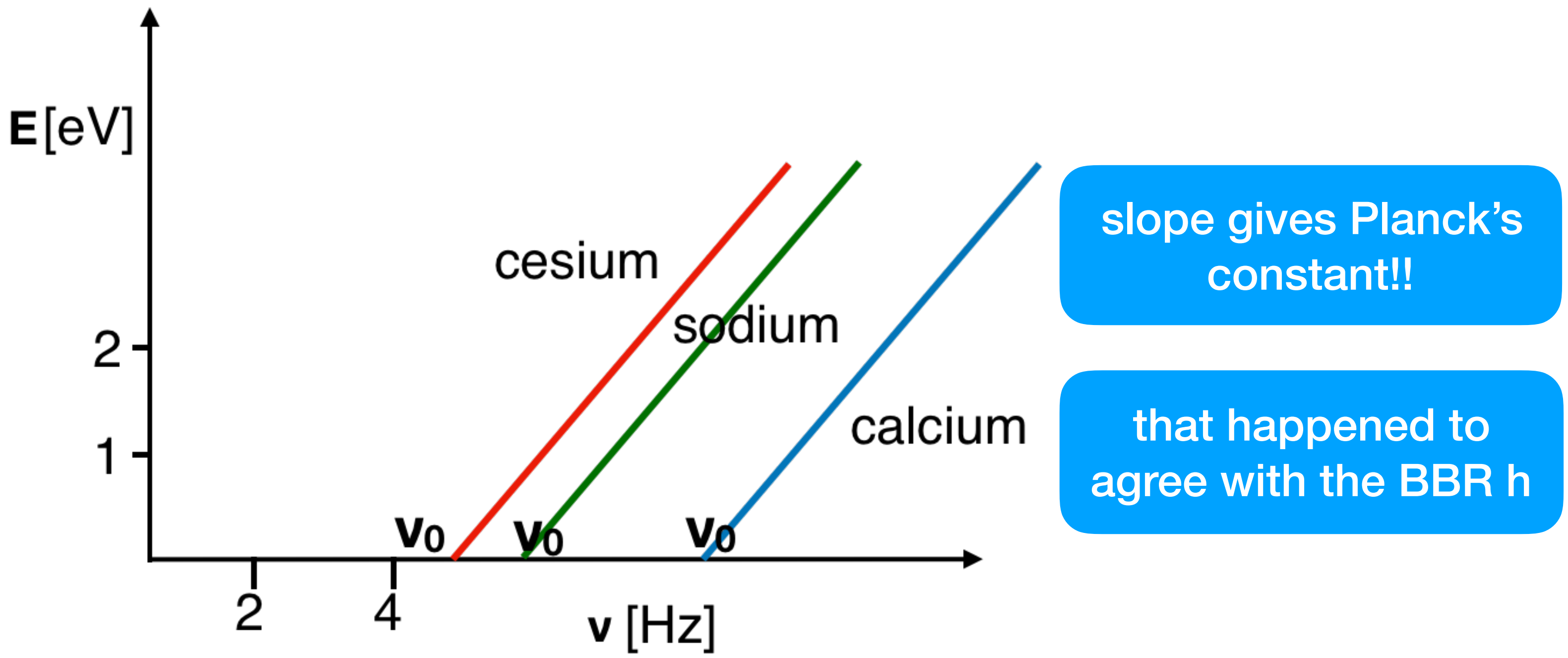


Photo-electric effect

max. **Kinetic energy of photo-electron**

$$KE_{max} = h\nu - \Phi \quad (29)$$

- Describes linear lines in experiment well:



- Can see that we need $\Phi = h\nu_0$ (30)

2.2.3) Wave-particle duality of light

We can now combine section 2.2.1), 2.2.2):

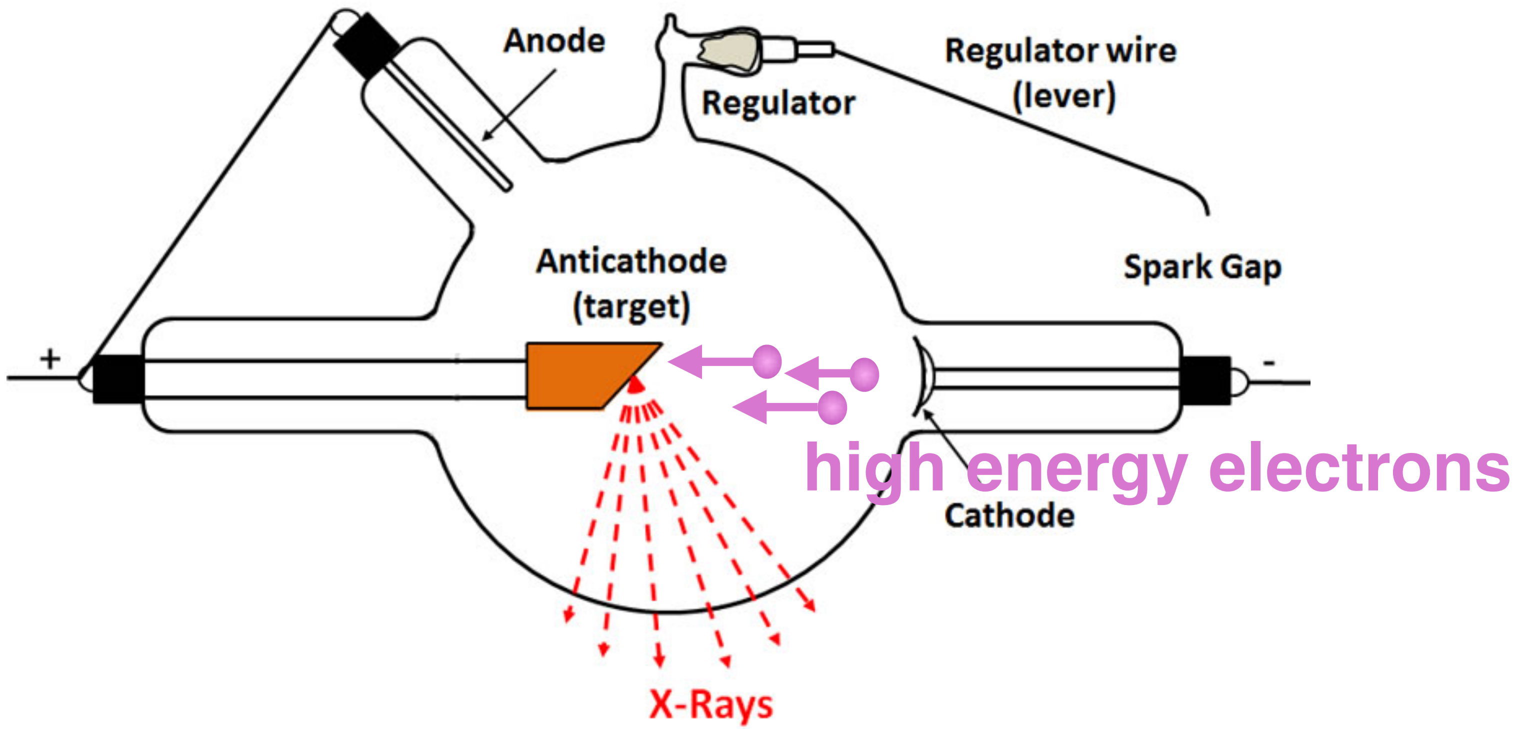
Light (electro-magnetic waves) are a **quantum field** of **photons** with mass=0.

- Quantum field = field **(E,B)** can only change in discrete quanta (due to at least one photon)
- propagates like waves **(interference / diffraction)**
- Created and destroyed like **particles**

2.2.4) X-rays

photo-effect: photon gives its whole energy to electron. What with the reverse?

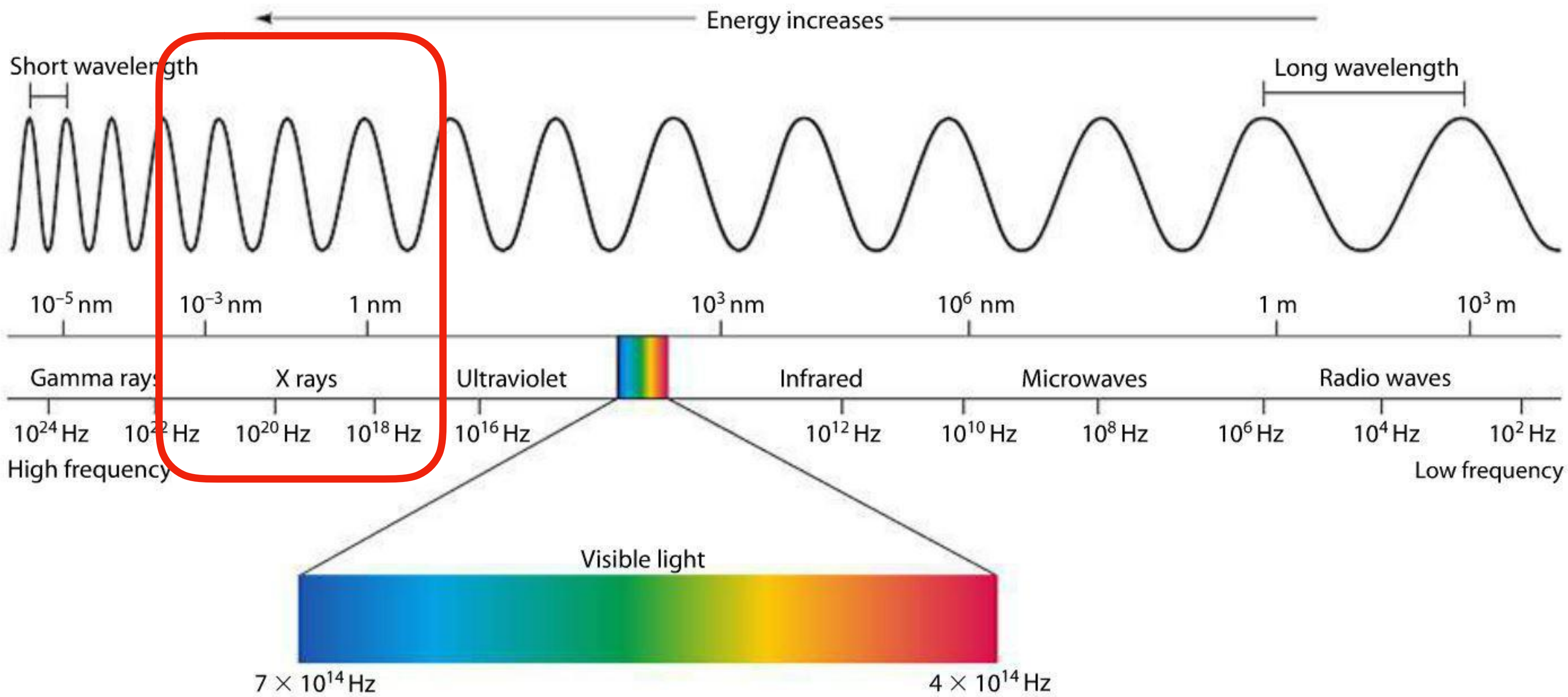
Inverse experiment: X -ray tube



https://www.orau.org/ptp/collection/xraytubes/introduction.htm

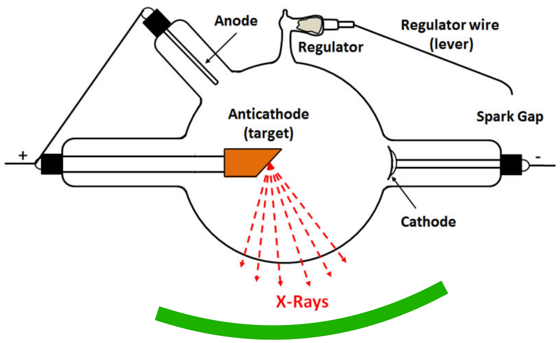
X-rays

Such experiments found new type of radiation called **X-rays**, much shorter λ than even UV

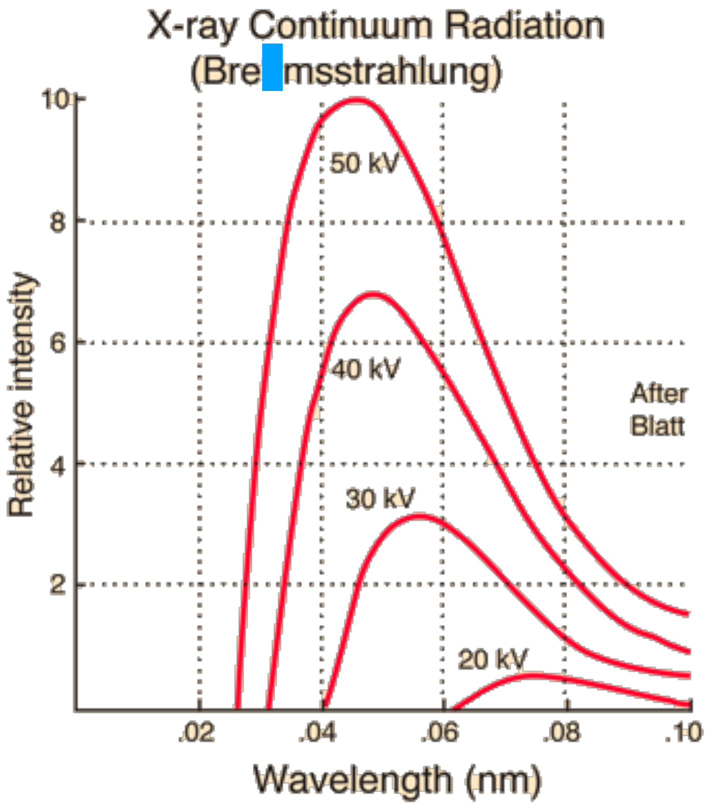


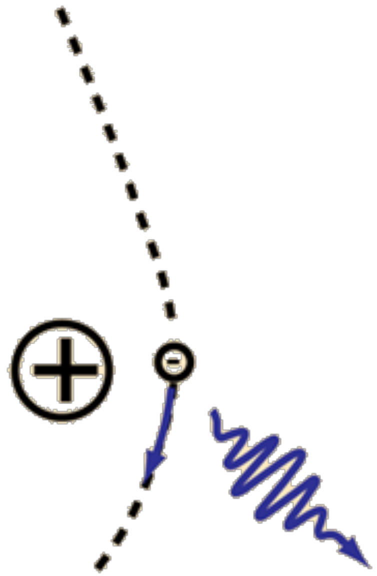
X-ray spectra

Bremsstrahlung (braking radiation)



λ ?



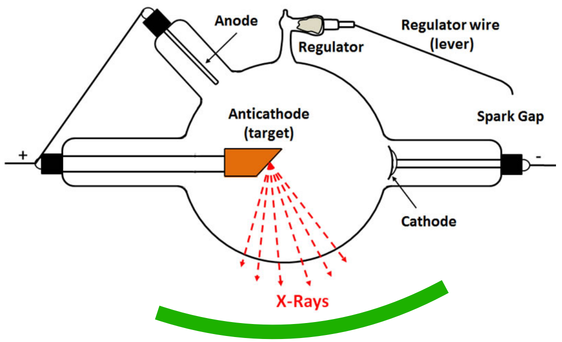


Accelerated electron emits radiation

same for all target materials

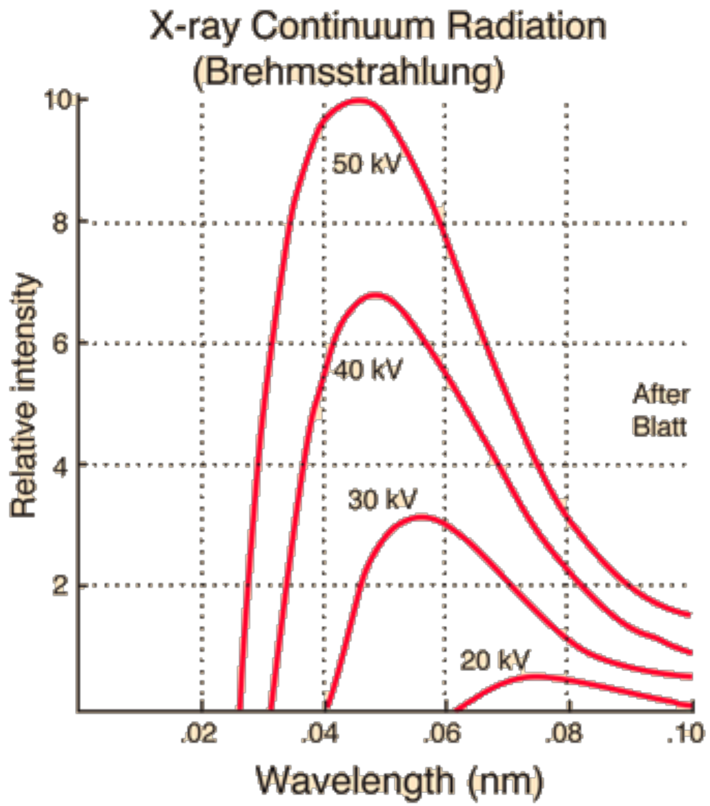
http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/xrayc.html

X-ray spectra



λ ?

Highest energy X-ray has **all the** electrons energy



Use Eq. (28) and Eq. (10):

$$E = h\nu \qquad \nu\lambda = c$$

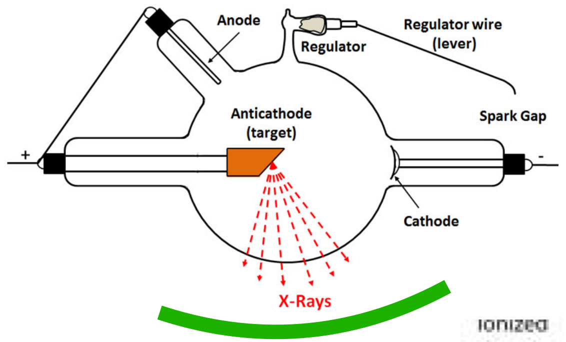
$$E = e\,Voltage = h\nu_{max} = \frac{hc}{\lambda_{min}}$$

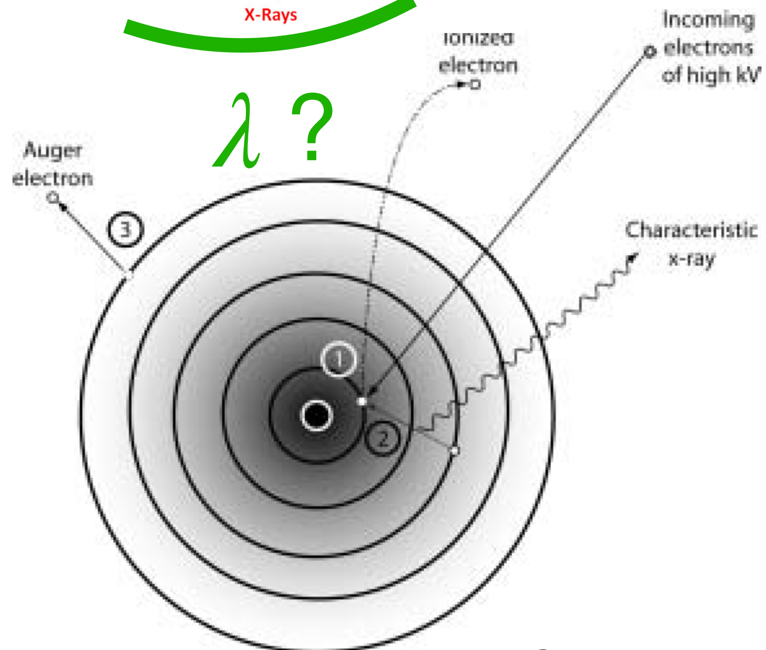
Minimal X-ray wavelength

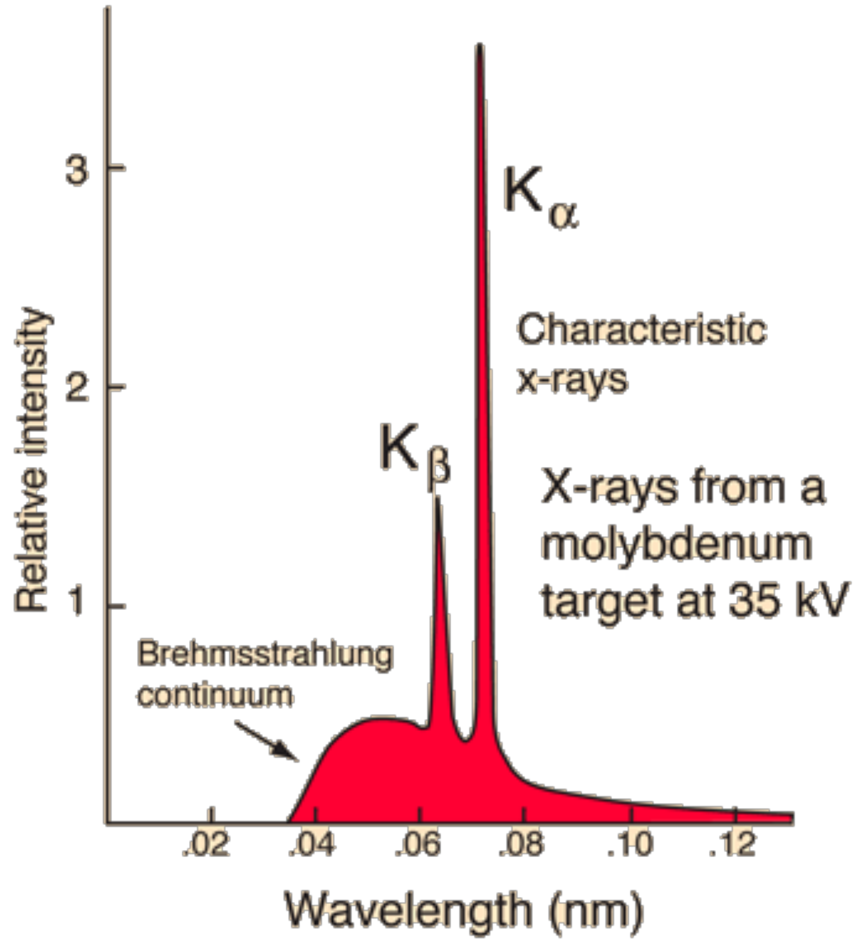
$$\lambda_{min} = \frac{hc}{e\,Voltage} \qquad \textbf{(31)}$$

X-ray spectra

Characteristic peaks on top of Bremsstrahlungs background



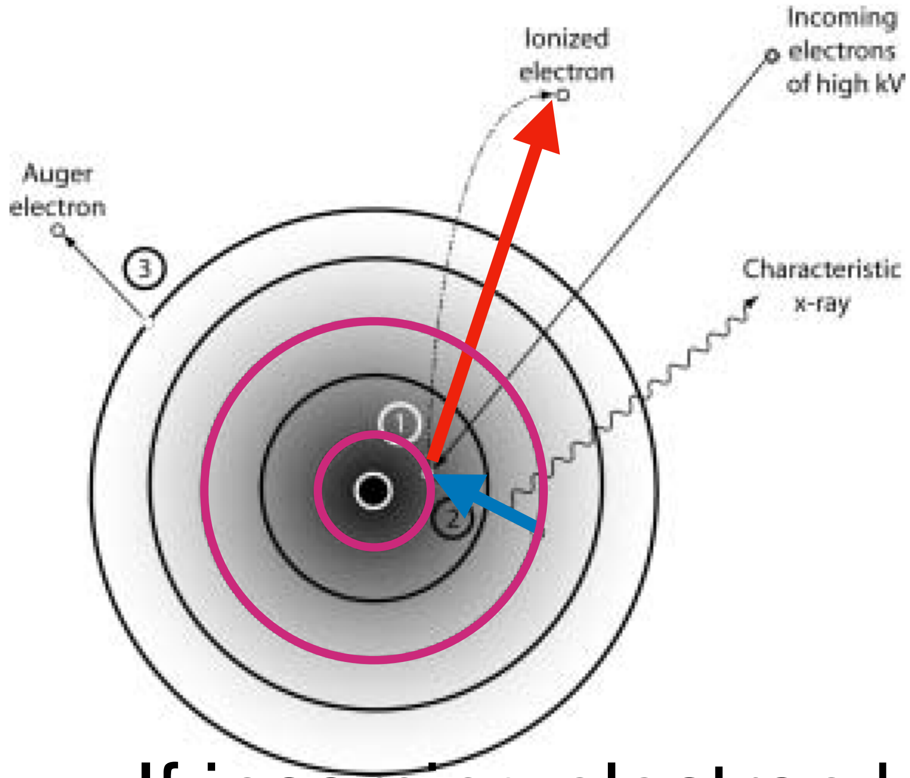




depend on target material

quantum states of electrons in atoms (week 7/10)

X-ray spectra



Electrons in atom can only have some specific energies (circular lines left)

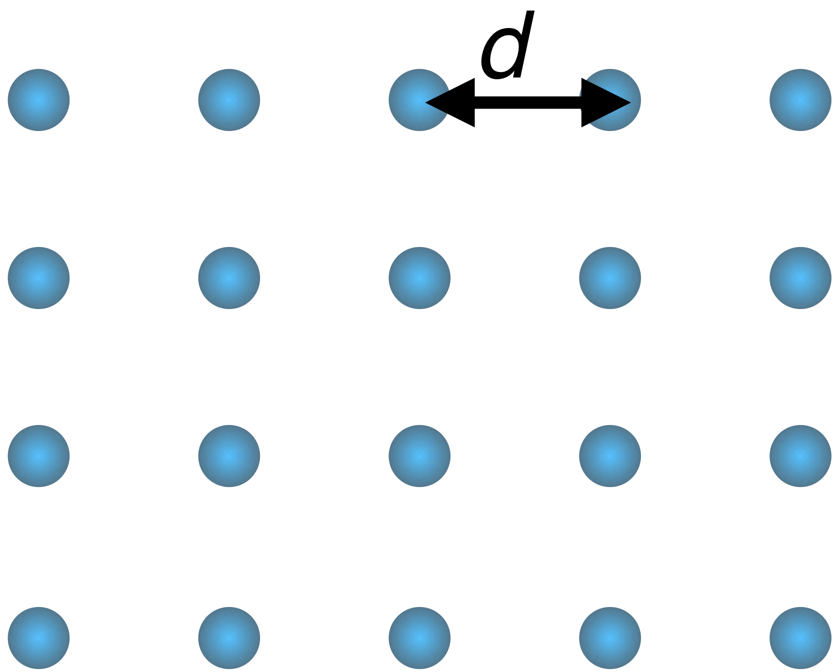
If incoming electron knocks out one of those (red), higher energy electrons (outer circle) “fall down” (blue)

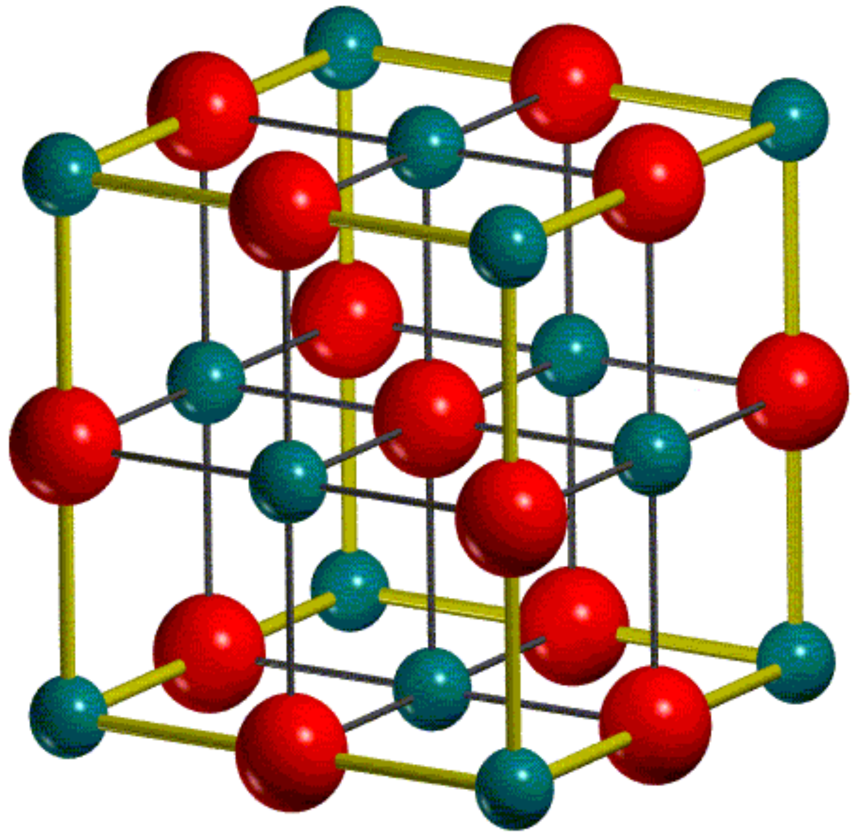
When they do that, they emit an X-ray photons, the wavelength (energy) of which precisely matches the energy difference between the two electronic states (violet lines)

X-ray diffraction

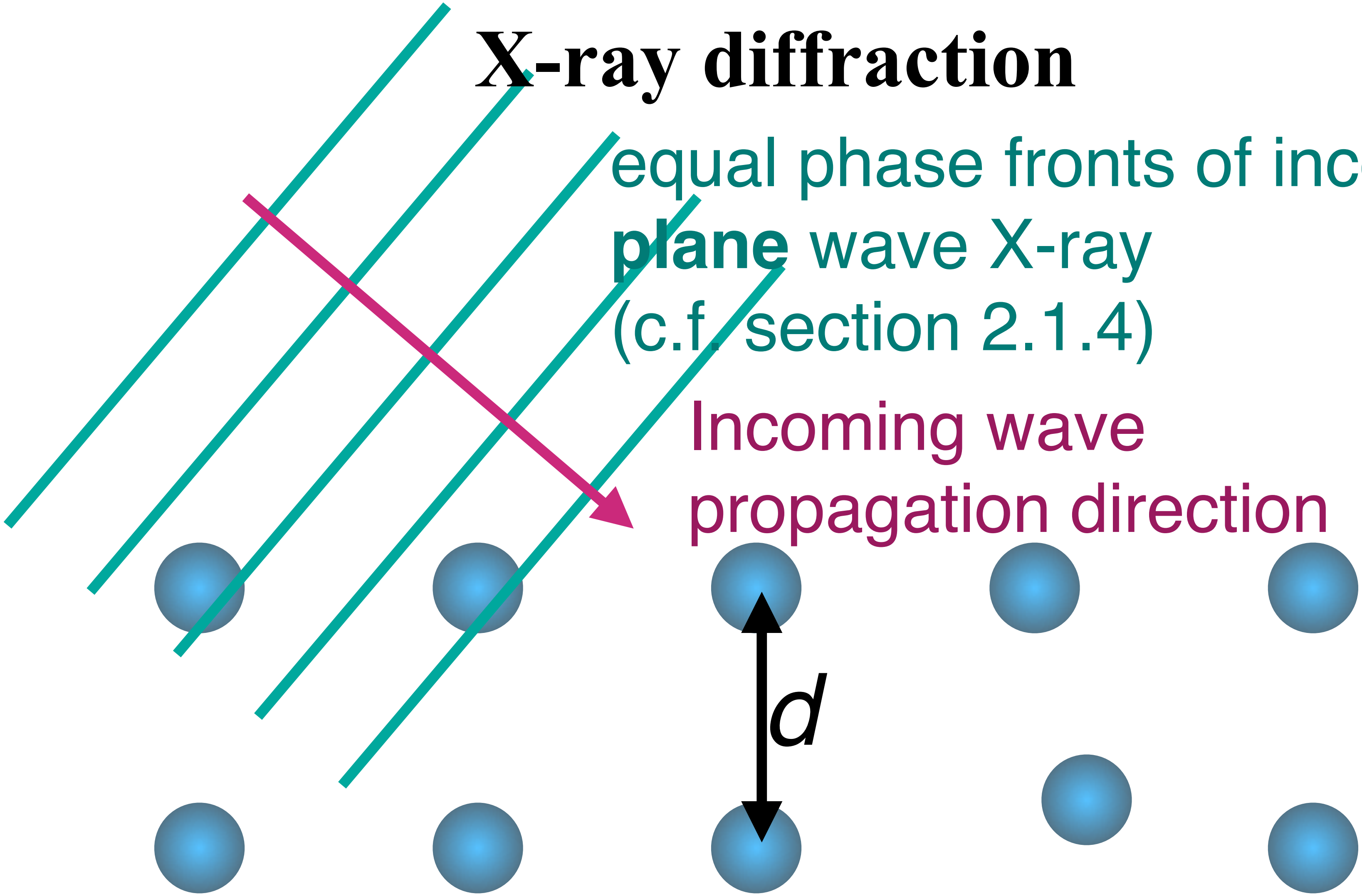
X-ray λ and solid crystal lattice constant *d* are both ~*nm*

Makes X-rays suitable to study crystals, and the reverse



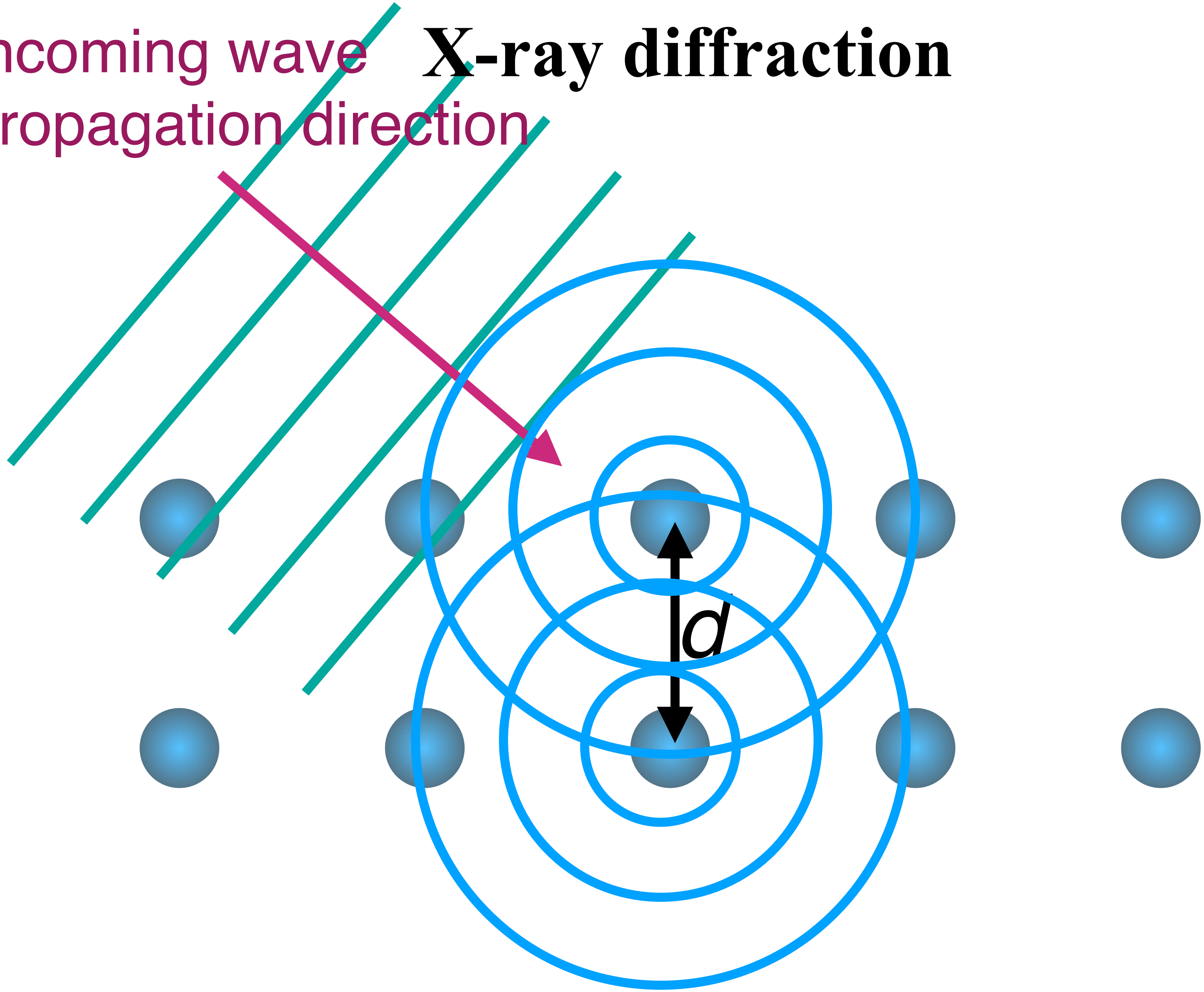


X-ray diffraction



X-ray diffraction

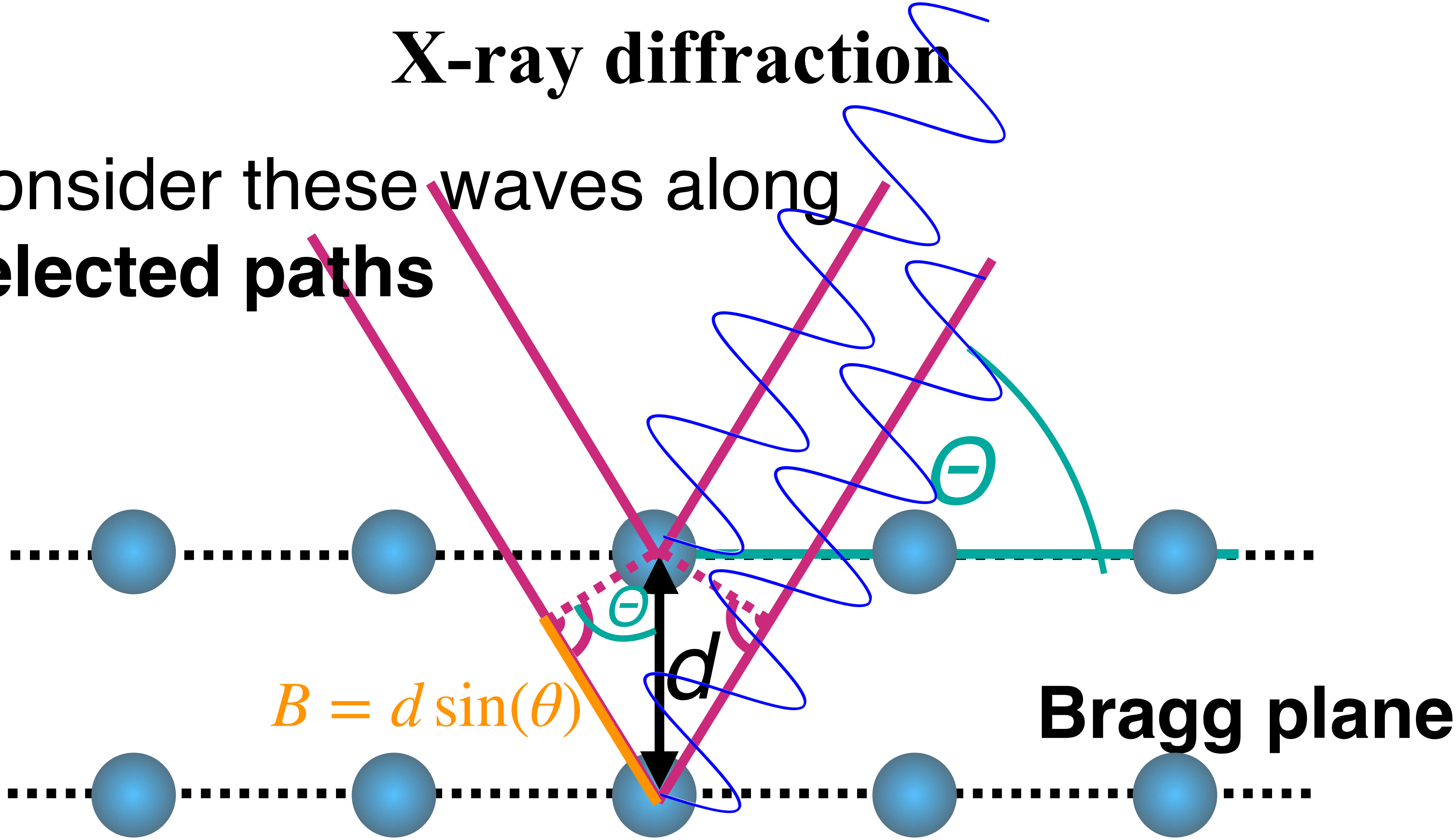
Incoming wave
propagation direction



Each crystal atom causes outgoing **spherical** scattered wave

X-ray diffraction

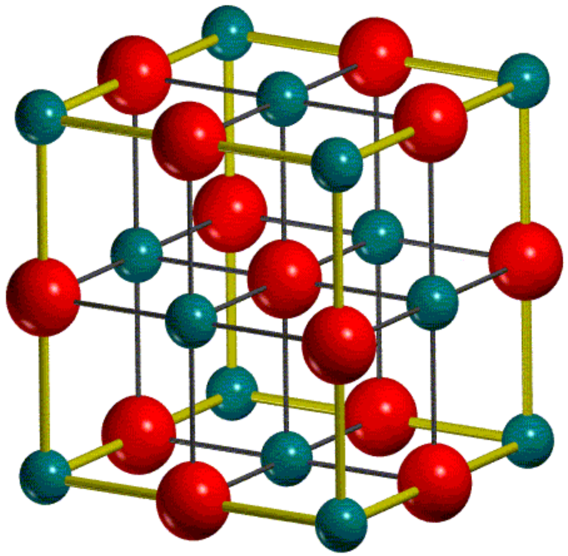
Consider these waves along **selected paths**



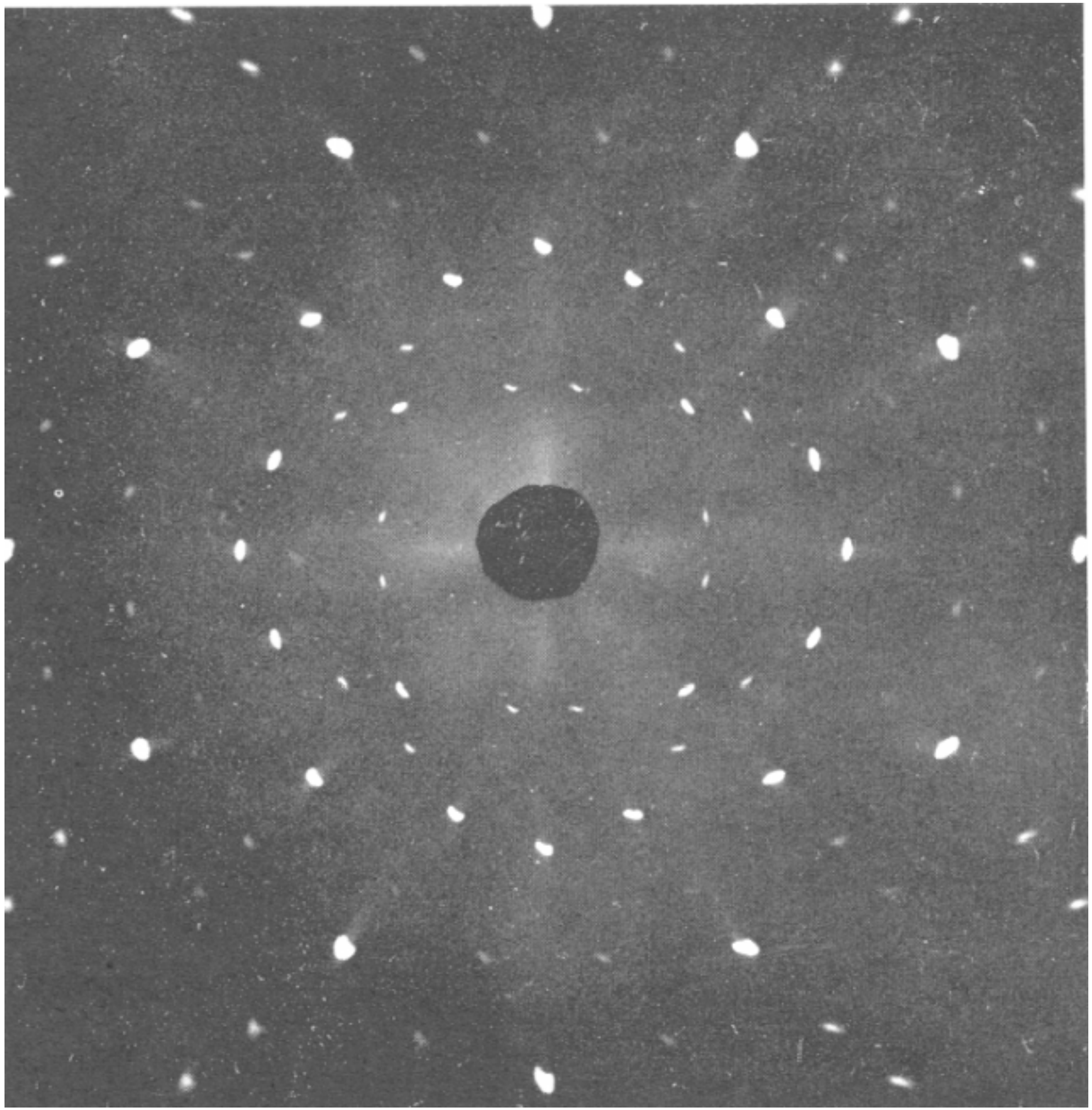
For constructive interference $2B = n\lambda \quad n = 1,2,3,\ldots$

$$2d\sin(\theta) = n\lambda$$

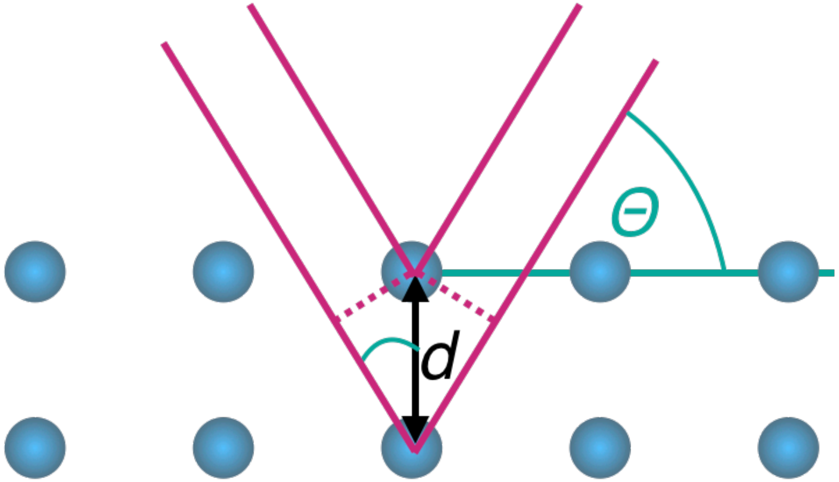
X-ray diffraction pattern

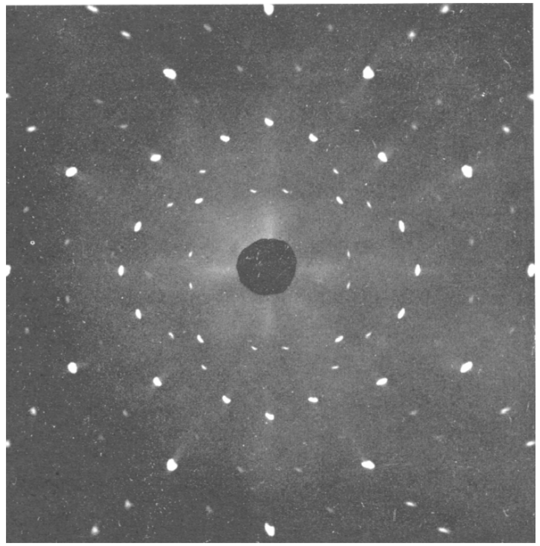


Many possible Bragg planes in 3D: complex structure



X-ray diffraction





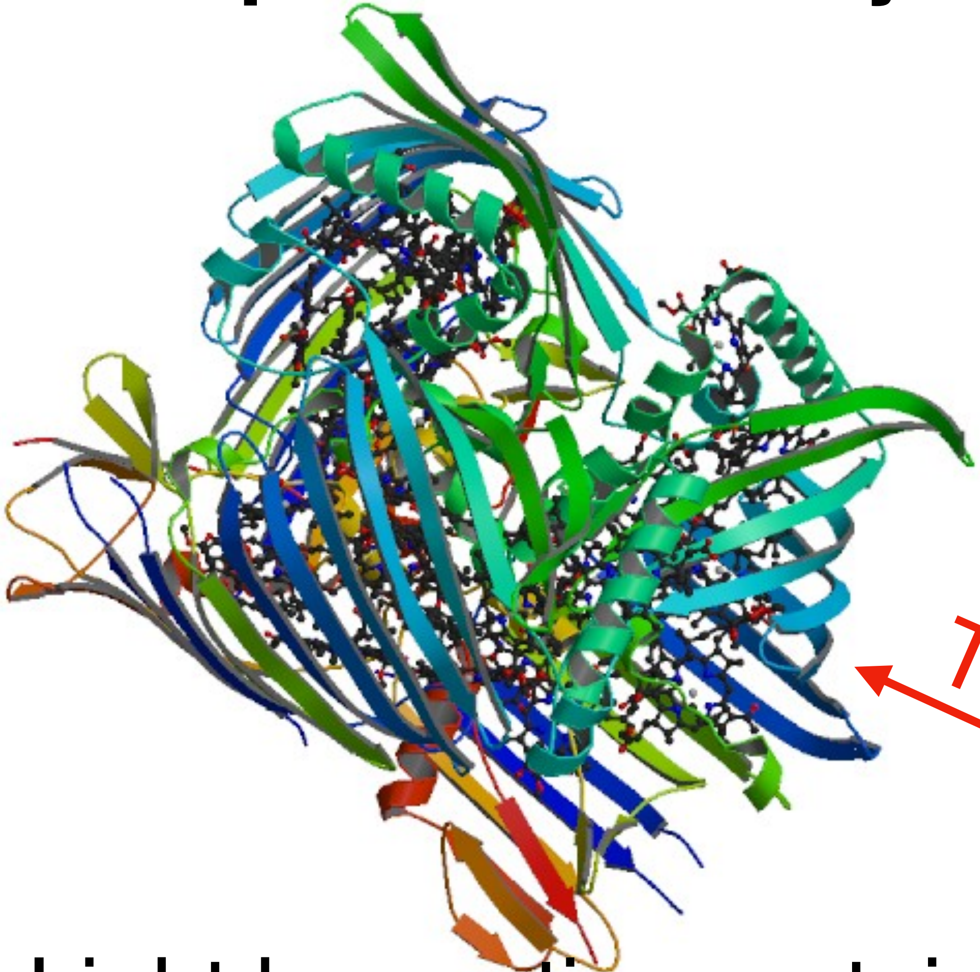
Bragg crystal diffraction **formula**

$$2d\sin(\theta) = n\lambda \qquad (32)$$

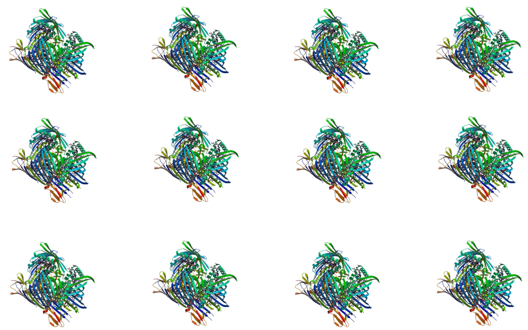
- for angles of constructive interference (spots)
- Find *d* from known *λ* or the reverse

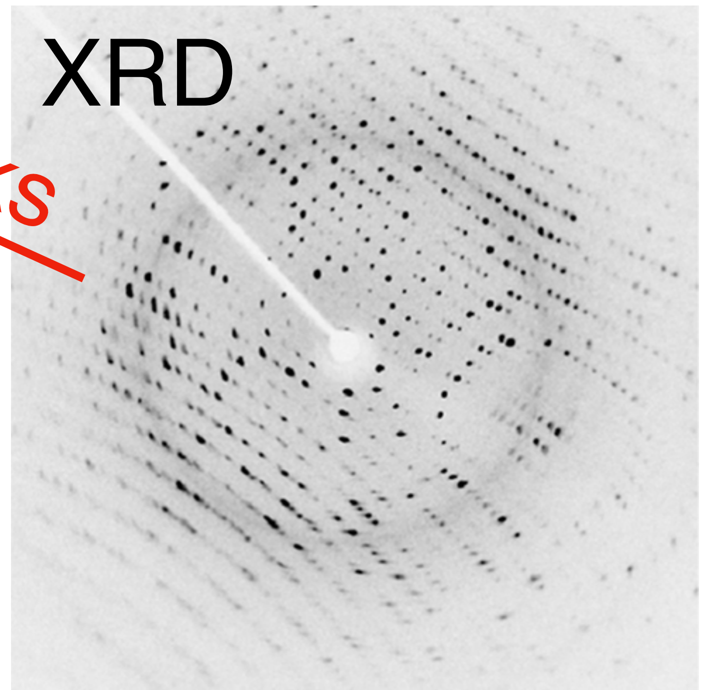
X-ray diffraction (XRD)

Example: Protein crystallography



Crystallize





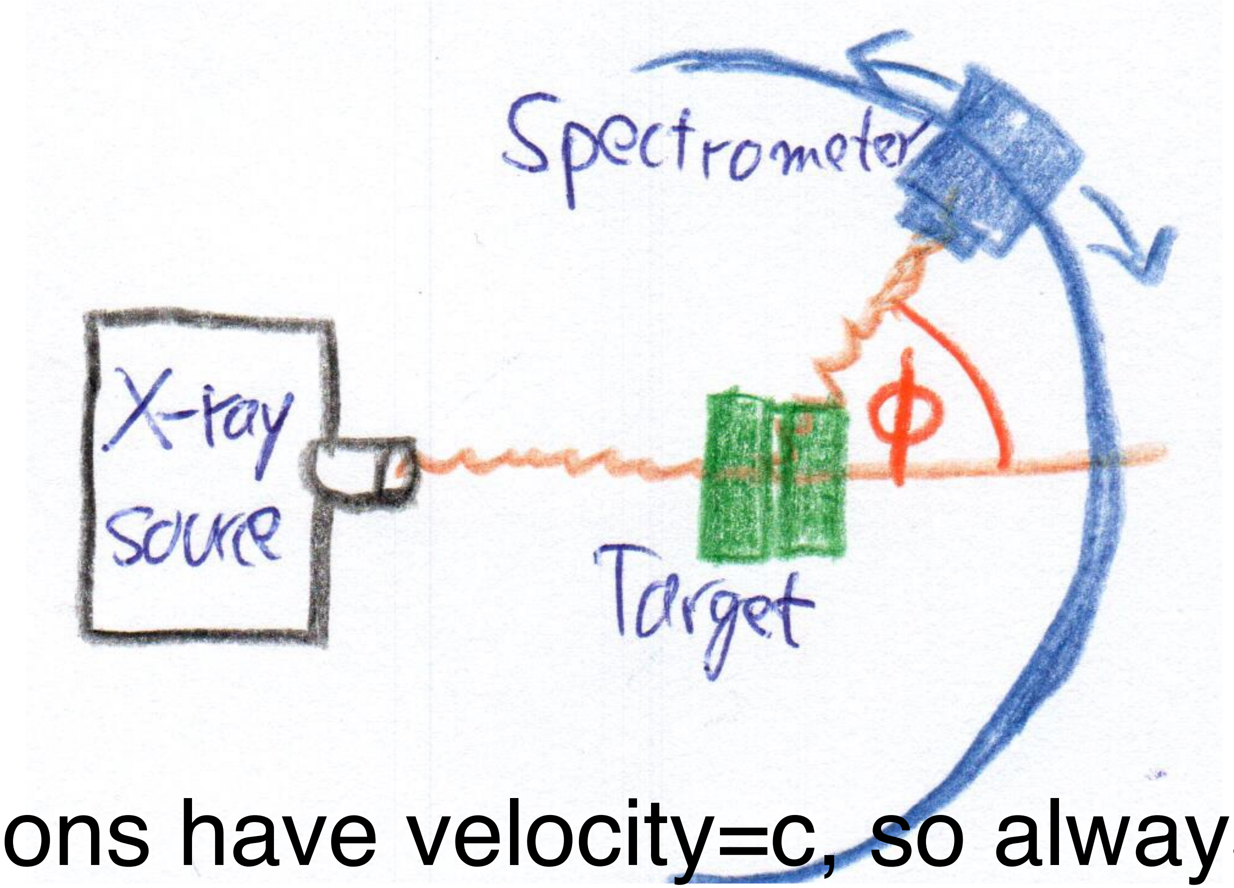
Tricks

Light harvesting protein

https://www.rcsb.org/structure/3ENI

2.2.5) Compton effect

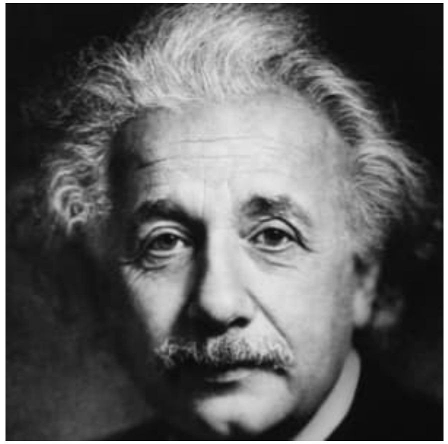
Photons behave like particles on emission, and absorption, also in scattering?



Photons have velocity=c, so always need relativity for their scattering….

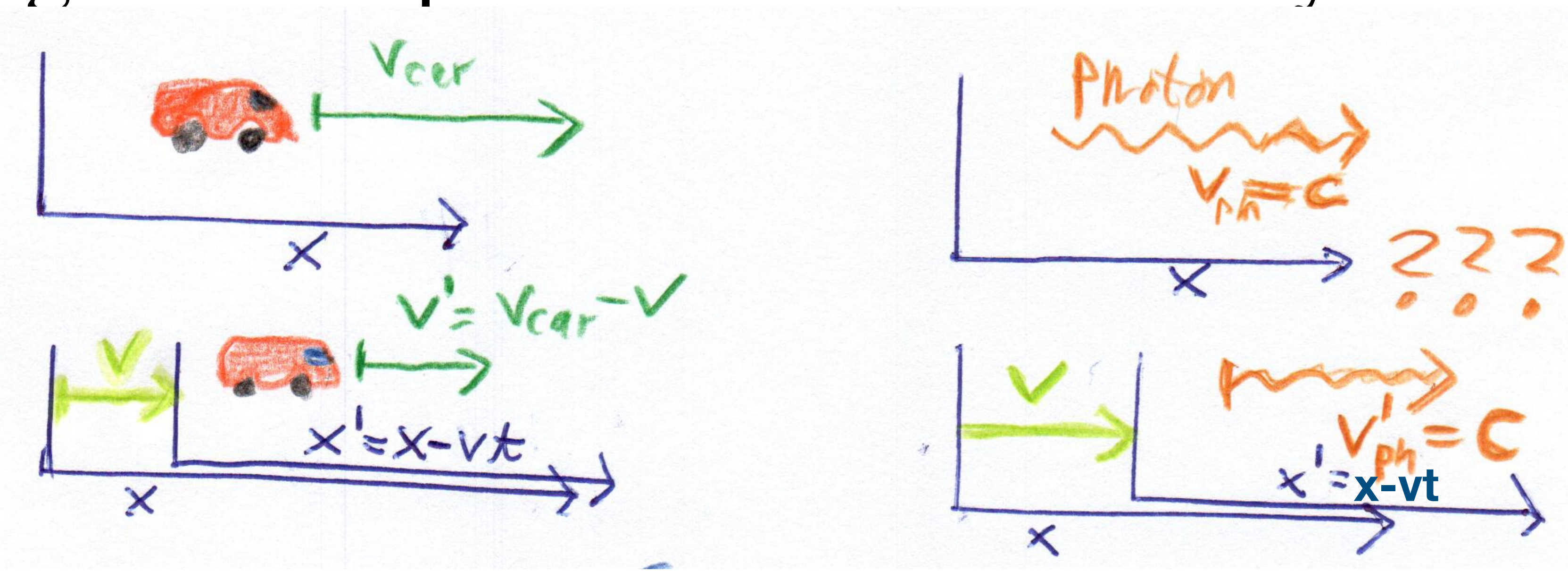
Mini excursion into Relativity

Experiment by Michelson & Morley: There is **no medium** for electro-magnetic waves



$\Rightarrow$

Core premise of special relativity: Speed of light is independent of coordinate system



Mini excursion into Relativity

Some consequences

Time dilation, time is **relative**

$$t' = \gamma t_0 \qquad \text{(33a)}$$

Observer moving at vel **v**, relative to

Lorentz factor: $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ (34)

Length contraction

$$L' = L_0/\gamma \qquad \text{(33b)}$$

more: e.g. Beiser book, chapter one

Mini excursion into Relativity

Consequences of consequences

Relativistic energy

$$E = \gamma m c^2 \qquad (33)$$

Hence:

speed limit $v < c$ if $m>0$ $v = c$ if $m=0$.

Relativistic momentum

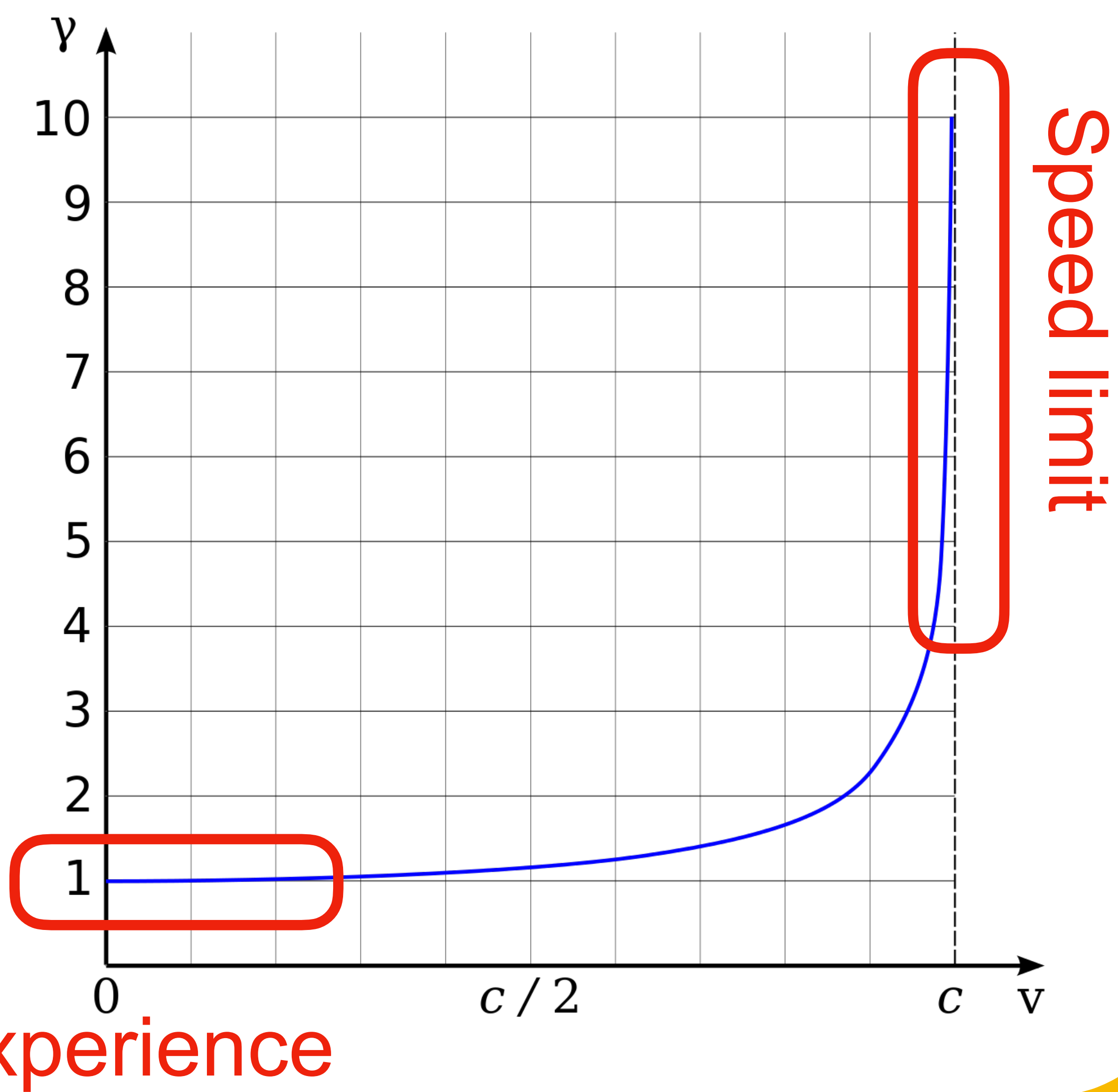
$$p = \gamma m v \qquad (35)$$

Mini excursion into Relativity

Speed limit $v < c$ if $m>0$ $v = c$ if $m=0$.

Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad (34)$$



Mini excursion into Relativity

Relativistic energy-momentum relation

$$E_{tot}^2 = p^2c^2 + (mc^2)^2 \qquad (36)$$

Rest energy

$$E_0 = mc^2 \qquad (37)$$

Kinetic energy

$$KE = E_{tot} - mc^2 \qquad (38)$$

Photon momentum

Momentum of massless particles (photon)

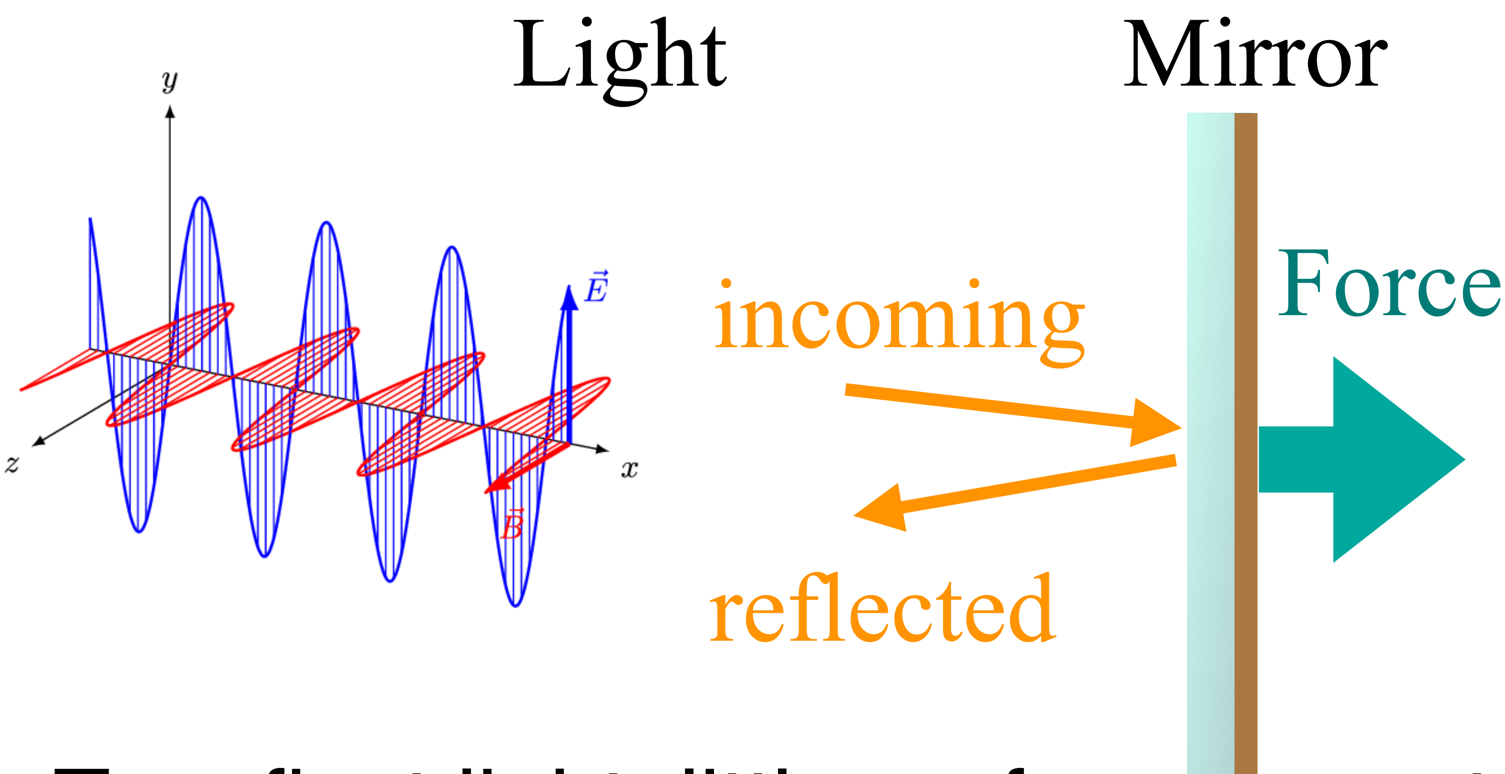
From Eq. (36)

$$E^2 = p^2c^2 + (mc^2)^2$$

$$E^2 = p^2c^2 \qquad p = E/c$$

Why does photon **have to** have a momentum?

Photon momentum

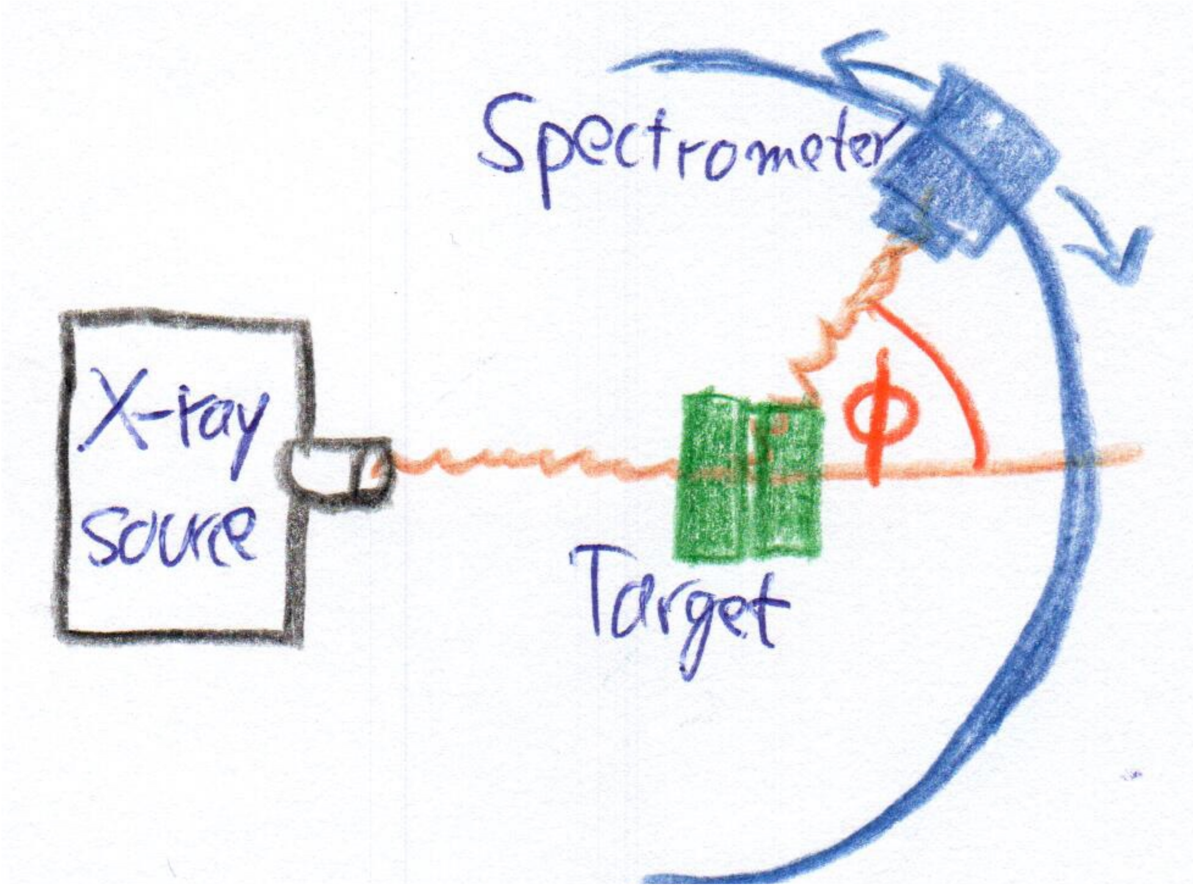


- To reflect light: little surface currents in mirror
- current+magnetic field = force = change of **momentum**

Application: solar cell space-craft

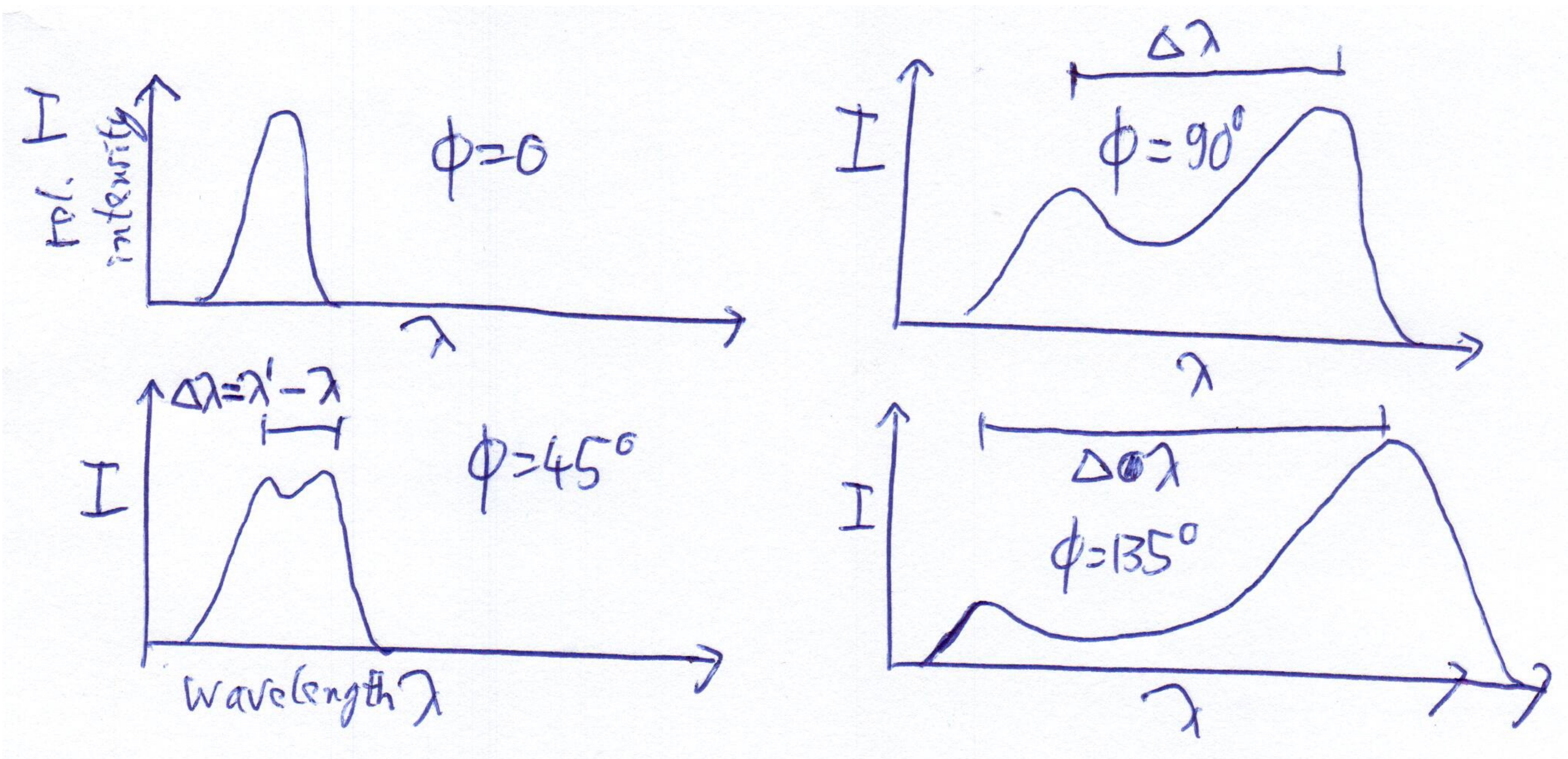
back to Compton effect

Photons behave like particles on emission, and absorption, also in scattering?



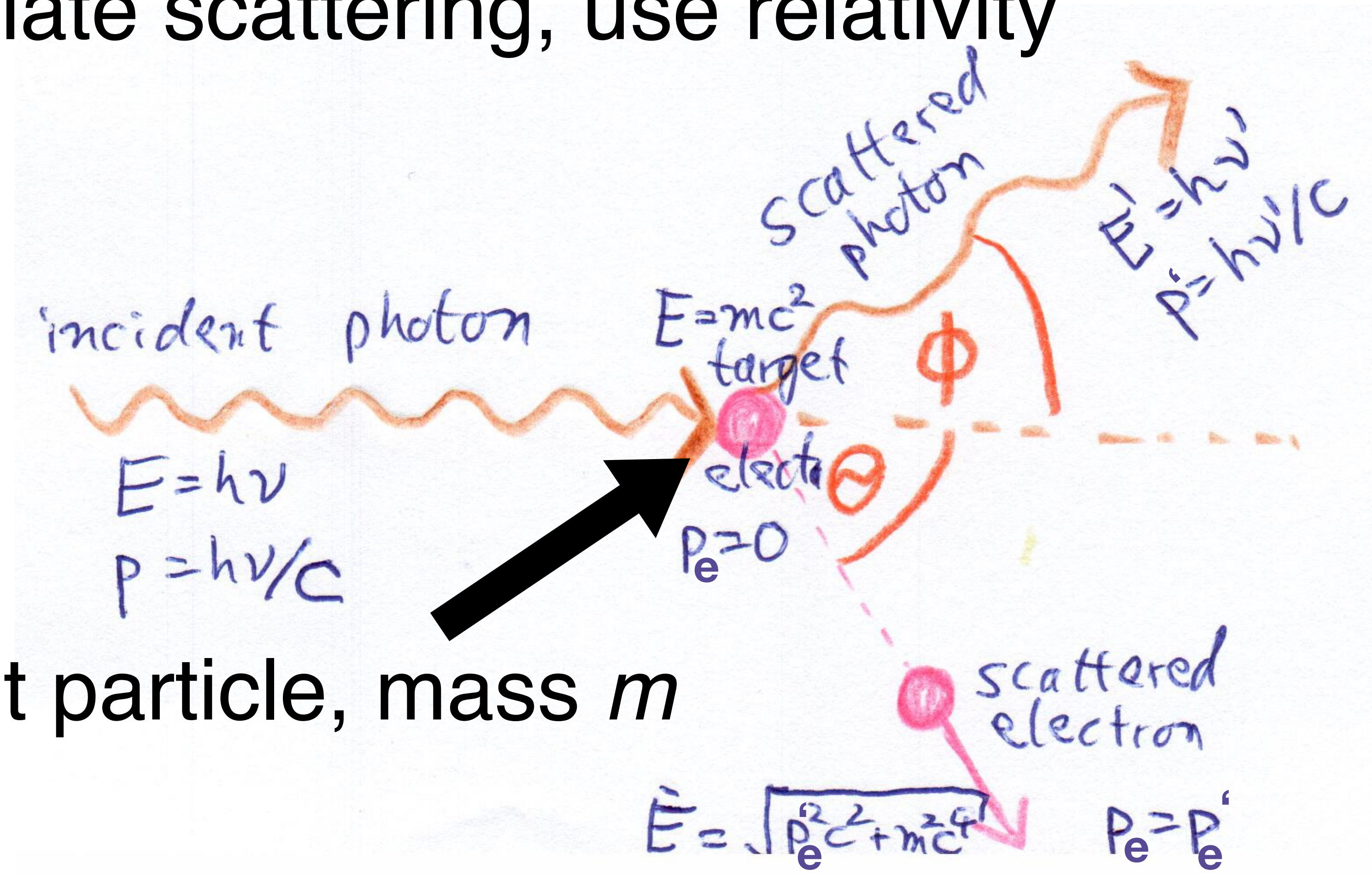
Compton effect

Measure wavelengths of scattered X-rays



Compton effect

Calculate scattering, use relativity



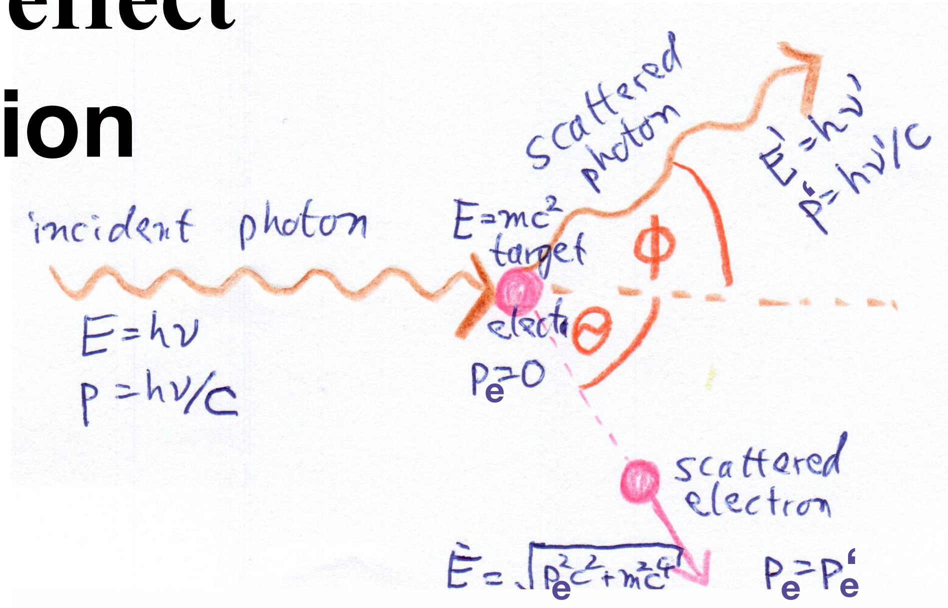
Target particle, mass *m*

Photon momentum (see Eq. (36))

$$p_{phot} = \frac{h\nu}{c} = \frac{h}{\lambda} \quad (39)$$

Compton effect

Momentum conservation



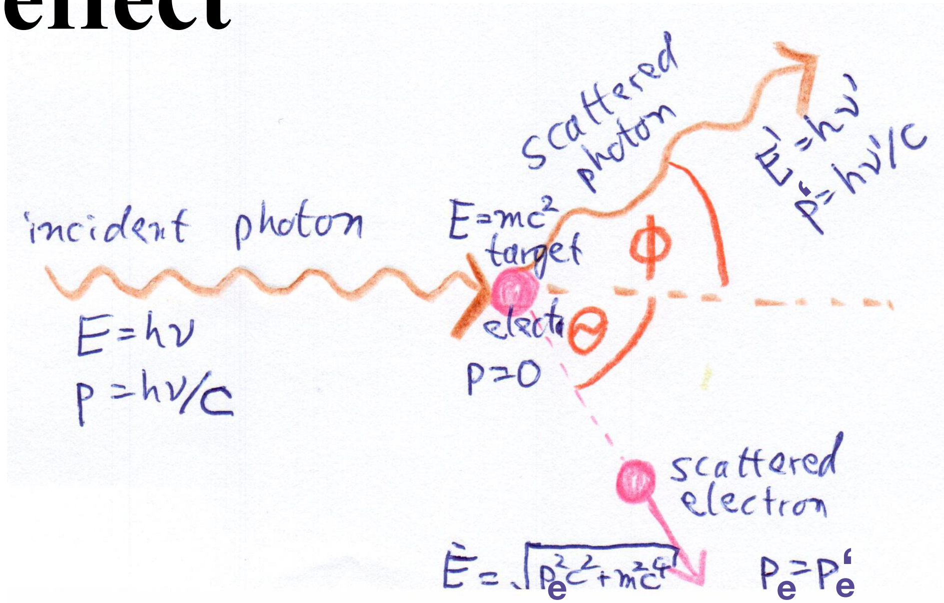
longitudinal momentum

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos(\phi) + p_e'\cos(\theta)$$

transverse momentum

$$0 = \frac{h\nu'}{c}\sin(\phi) - p_e'\sin(\theta)$$

Compton effect



longitudinal momentum

$$h\nu - h\nu' \cos(\phi) = p_e' c \cos(\theta) \qquad (\cdots)^2$$

$$+$$

transverse momentum

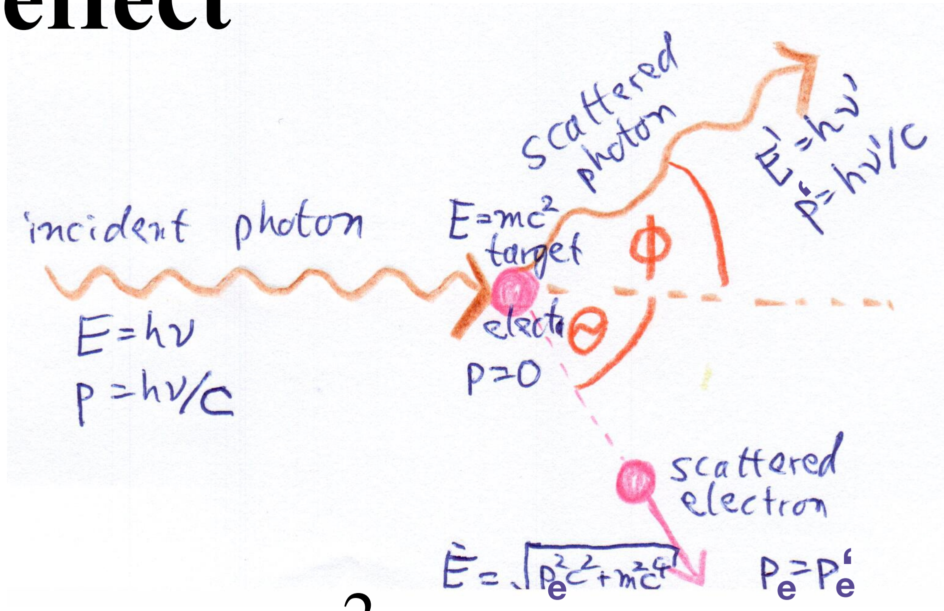
$$h\nu' \sin(\phi) = p_e' c \sin(\theta) \qquad (\cdots)^2$$

$$\Downarrow$$

$$p_e'^2 c^2 = (h\nu)^2 - 2(h\nu)(h\nu')\cos(\phi) + (h\nu')^2 \qquad \textbf{(40)}$$

Compton effect

Energy conservation



$$h\nu + mc^2 = h\nu' + KE + mc^2$$

$$KE = h\nu - h\nu'$$

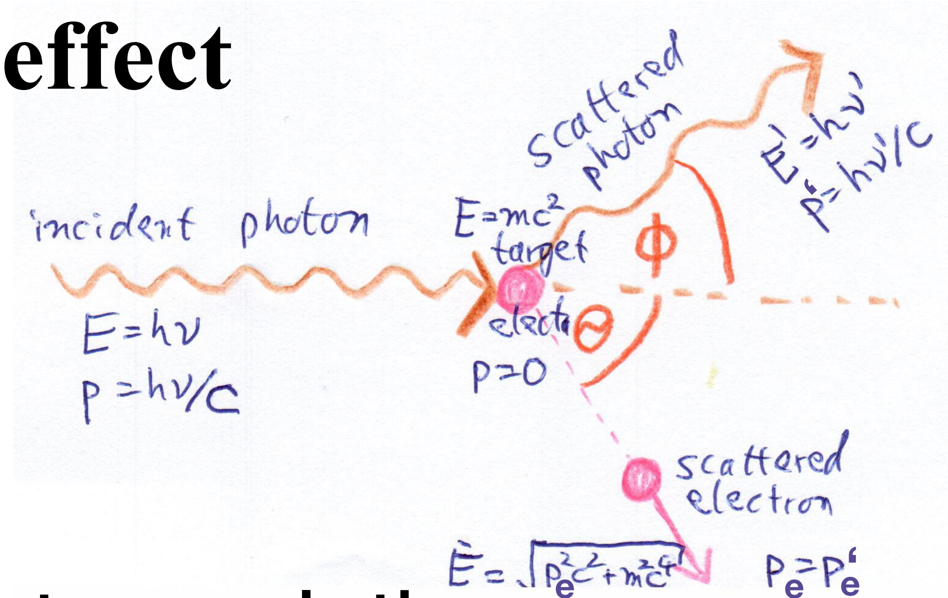
Relativistic energy-momentum relation

$$p_e'^2 c^2 = KE^2 + 2mc^2 KE$$

[from Eq. (36) and Eq. (38)]

$E^2 = p_e'^2 c^2 + (mc^2)^2$ $\qquad KE = E - mc^2$

Compton effect

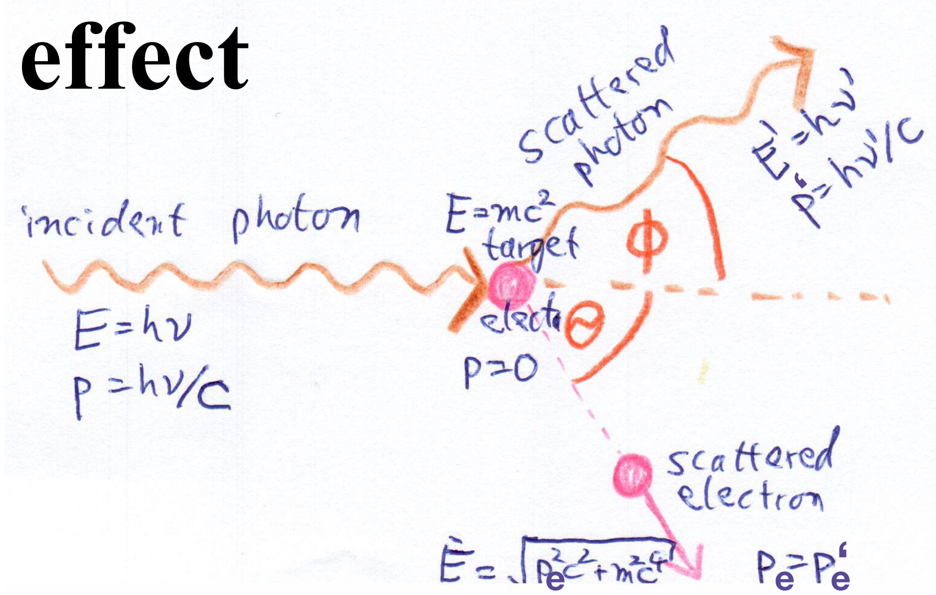


Relativistic energy-momentum relation

$$p_e'^2c^2 = (h\nu)^2 - 2(h\nu)(h\nu') + (h\nu')^2 + 2mc^2(h\nu - h\nu')$$

Insert Eq. (40) for $p_e'^2c^2$ and few more steps (see book)….

Compton effect



Compton scattering **formula**

$$\lambda' - \lambda = \lambda_C[1 - \cos(\phi)] \quad \textbf{(41)}$$

Compton wavelength of the electron (if target)

$$\lambda_C = \frac{h}{mc} = 2.426 \times 10^{-12} m \quad \textbf{(41b)}$$

- (41) predicts increased λ for larger angles ϕ

Compton effect

Compton scattering **formula**

$$\lambda' - \lambda = \lambda_C[1 - \cos(\phi)] \quad (41)$$

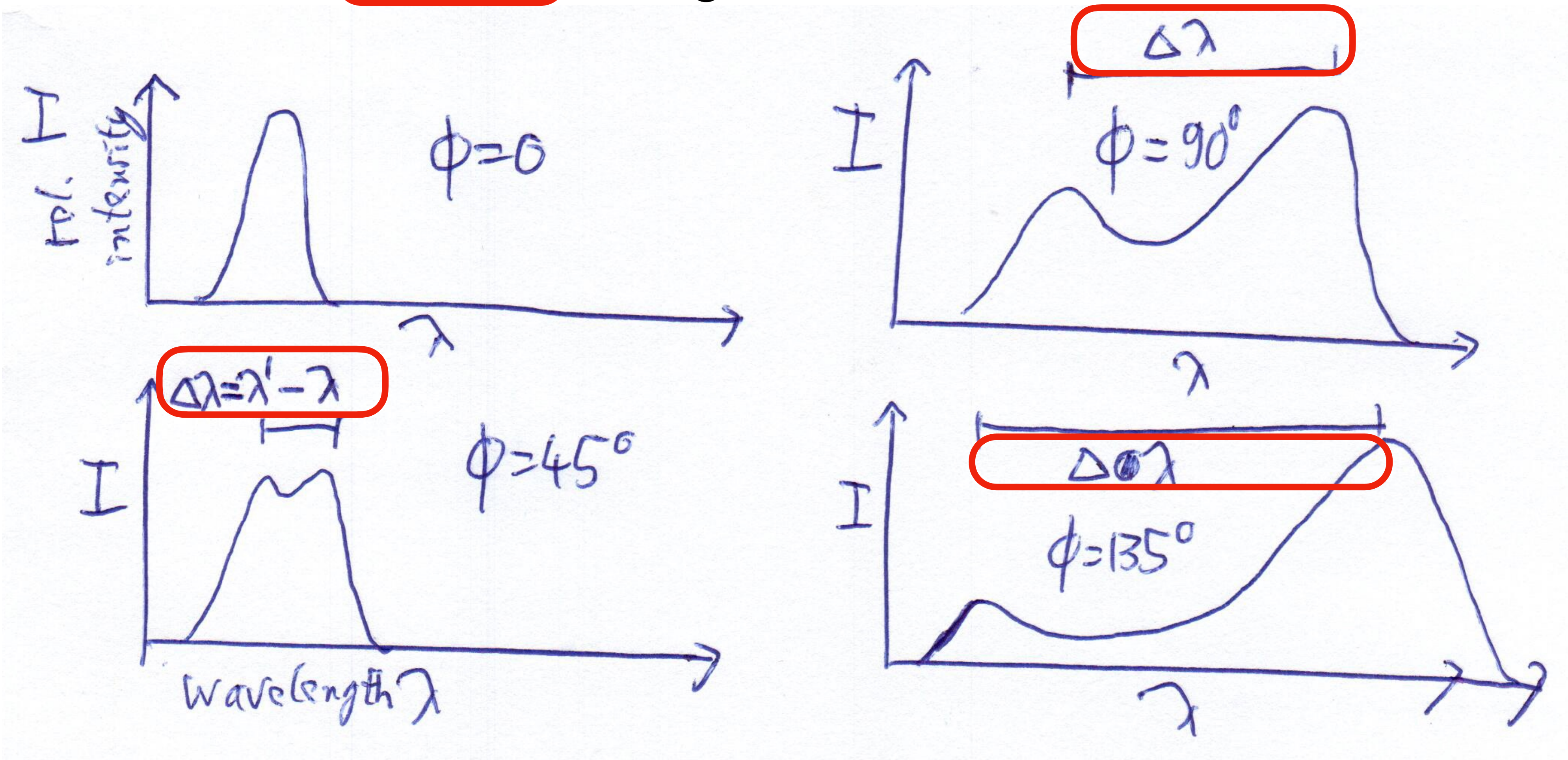
Compton wavelength of the electron

$$\lambda_C = \frac{h}{mc} = 2.426 \times 10^{-12} m \quad (41b)$$

- (41) predicts increased λ for larger angles ϕ
- Q: What spectral range is λ_C in? It is tiny, what does that mean?
- A: X-ray. Will not notice effect for softer light.

Compton effect

$$\lambda' - \lambda = \lambda_C[1 - \cos(\phi)]$$



Q: Why the unshifted part?

- A: See book. Scattering off rigid inner electrons

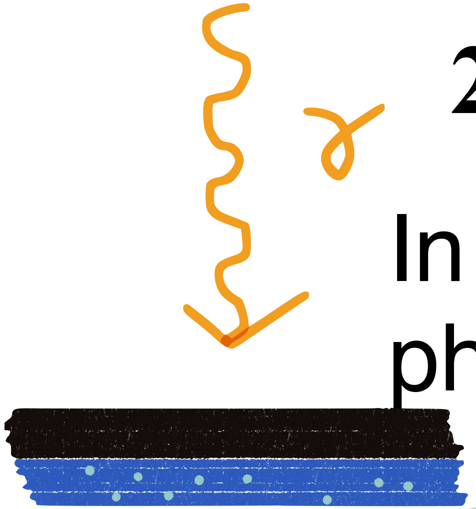
Compton effect

Spectrum has a part that shifts, and a part that does not seem to shift. Why?

- In the derivation of Eq. (41), we did not use the fact that the target was an electron.

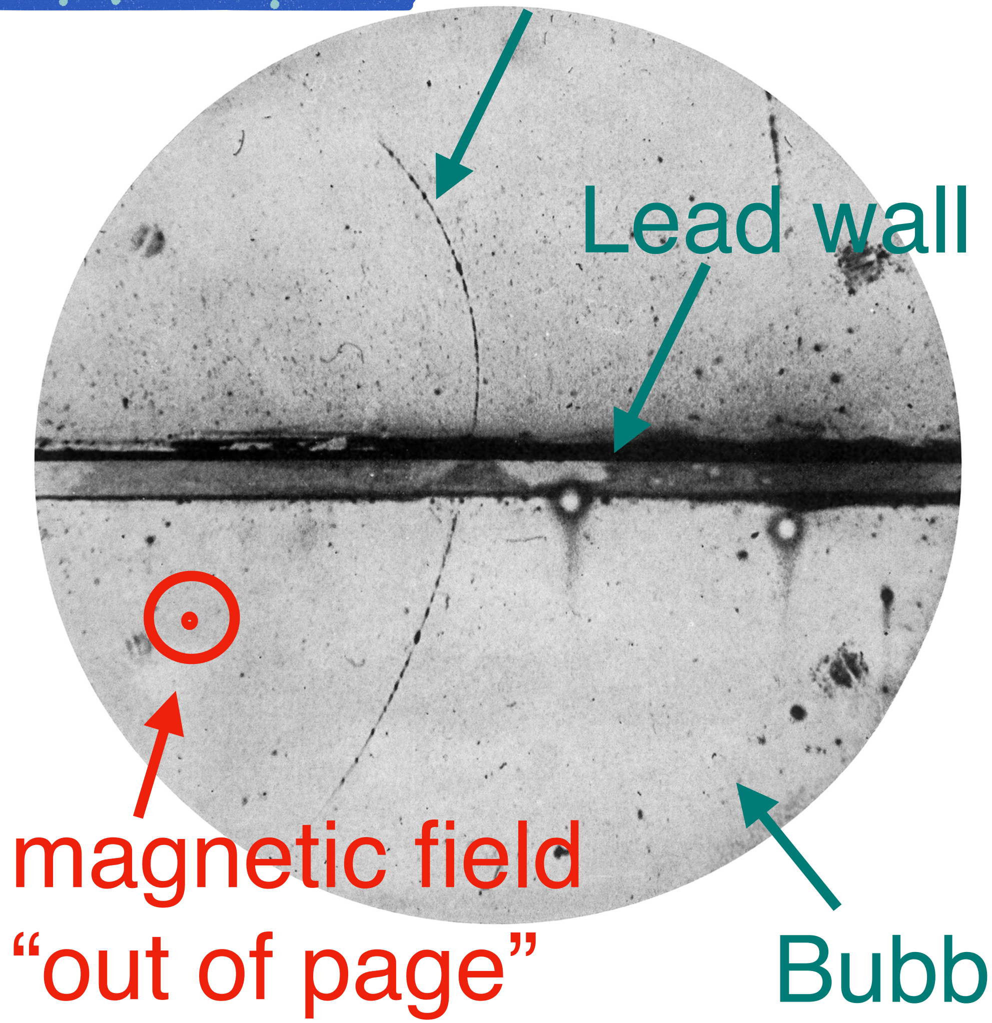
- Thus photon scattering off anything, would give same equation, with electron mass *m* in Eq. (41) replaced by other target mass

- If photon scatters of heavy object (nucleus, whole atom), mass is 1/1000 that of electron -> no visible shift.



2.2.7) Pair production, positrons

In the 1920/30s, while we were shooting photons onto things.....



- Let’s move from X-ray to cosmic γ-ray
- Inspect collision results via Lorentz Force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

(orthogonal to motion, direction dep. on charge sign!!!)

2.2.7) Pair production $E = mc^2$

Photon can turn into electron-positron pair

Positron is the **anti-particle** of the electron

- Same mass $m = m_e$ but opposite charge

$$q = +e > 0$$

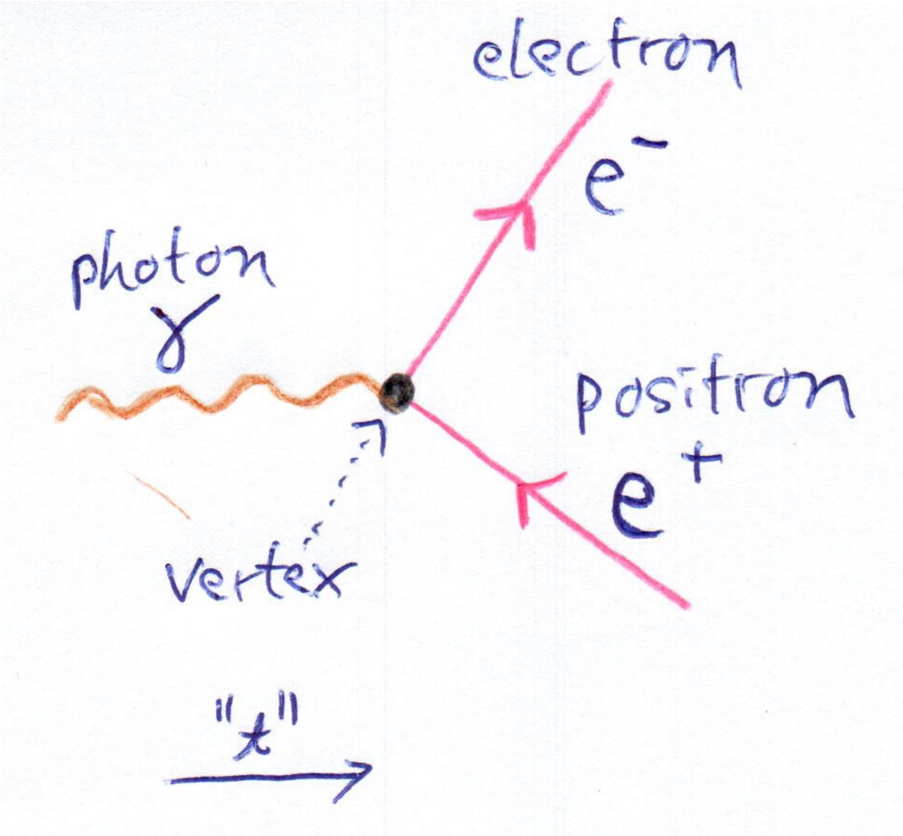
(Picture on previous slide: First observation of positron and thus anti-matter)

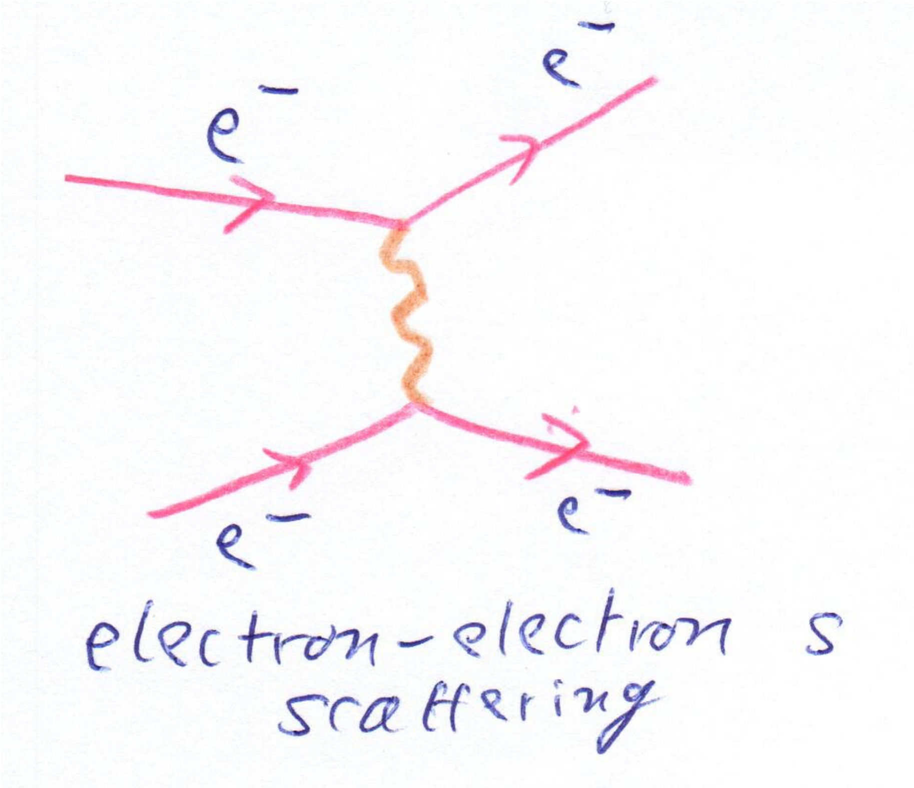
- Existence of anti-particles required in **relativistic quantum mechanics**

Pair production

Photon can turn into electron-positron pair

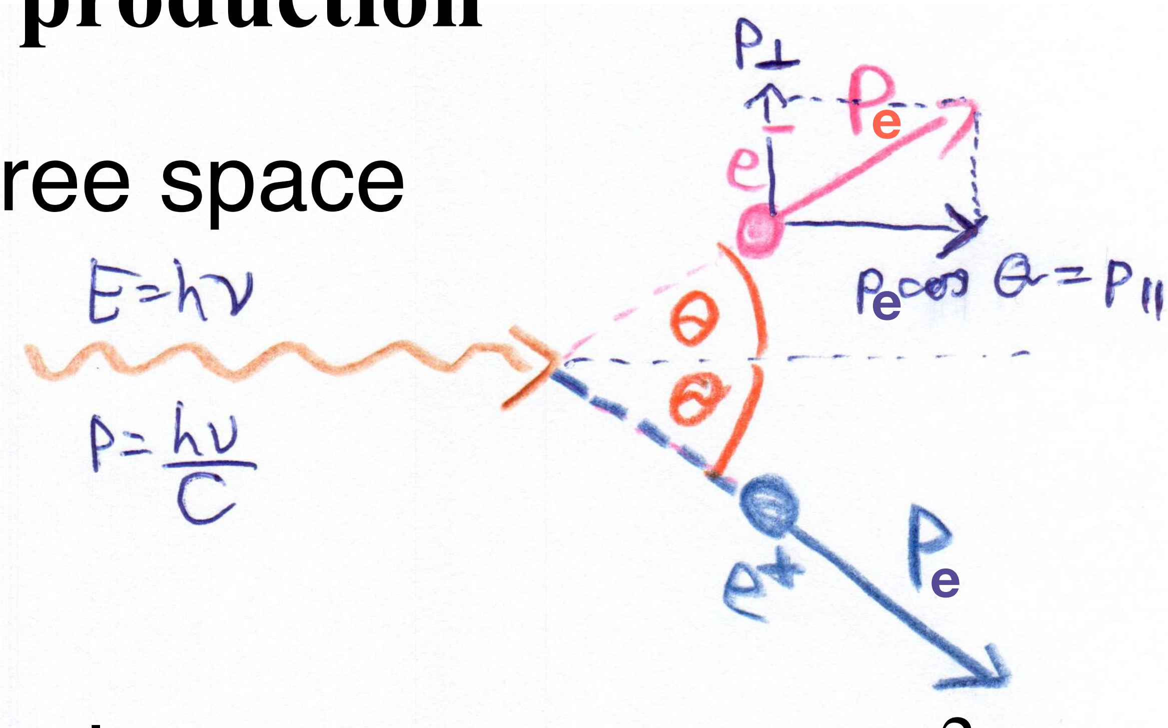
Positron is the **anti-particle** of the electron





Pair production

Cannot occur in free space



- Energy conservation: $$h\nu = 2\gamma mc^2$$

Need $h\nu > 2mc^2 = 2 \times 0.511$ MeV

Hence $\lambda < 7.62$ pm

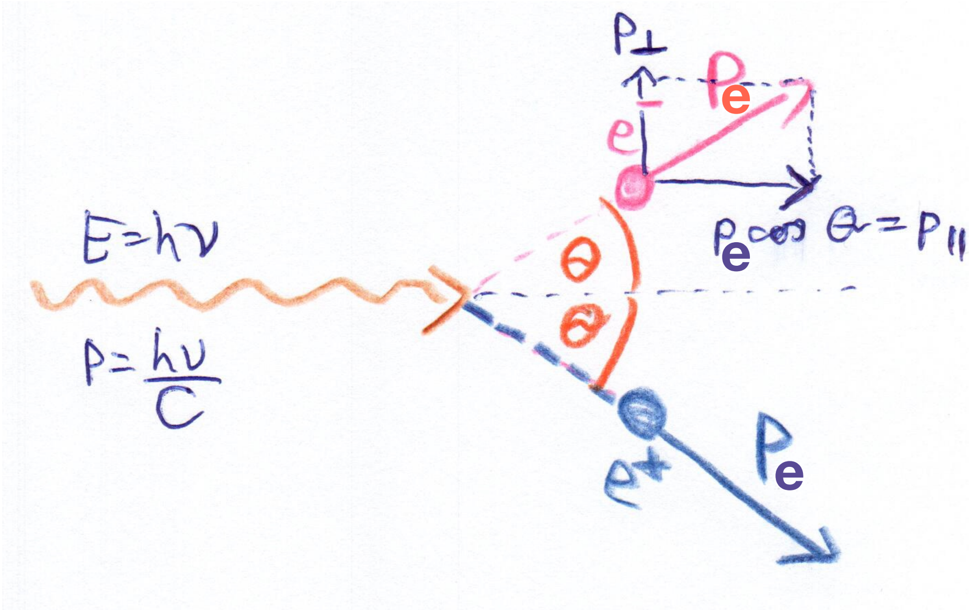
$$p_e = \gamma mv$$

- Momentum conservation: $$\frac{h\nu}{c} = 2p_e\cos(\theta)$$

Pair production

- Energy conservation: $h\nu = 2\gamma mc^2$

- Momentum conservation: $h\nu = 2(\gamma mv)c\cos(\theta)$

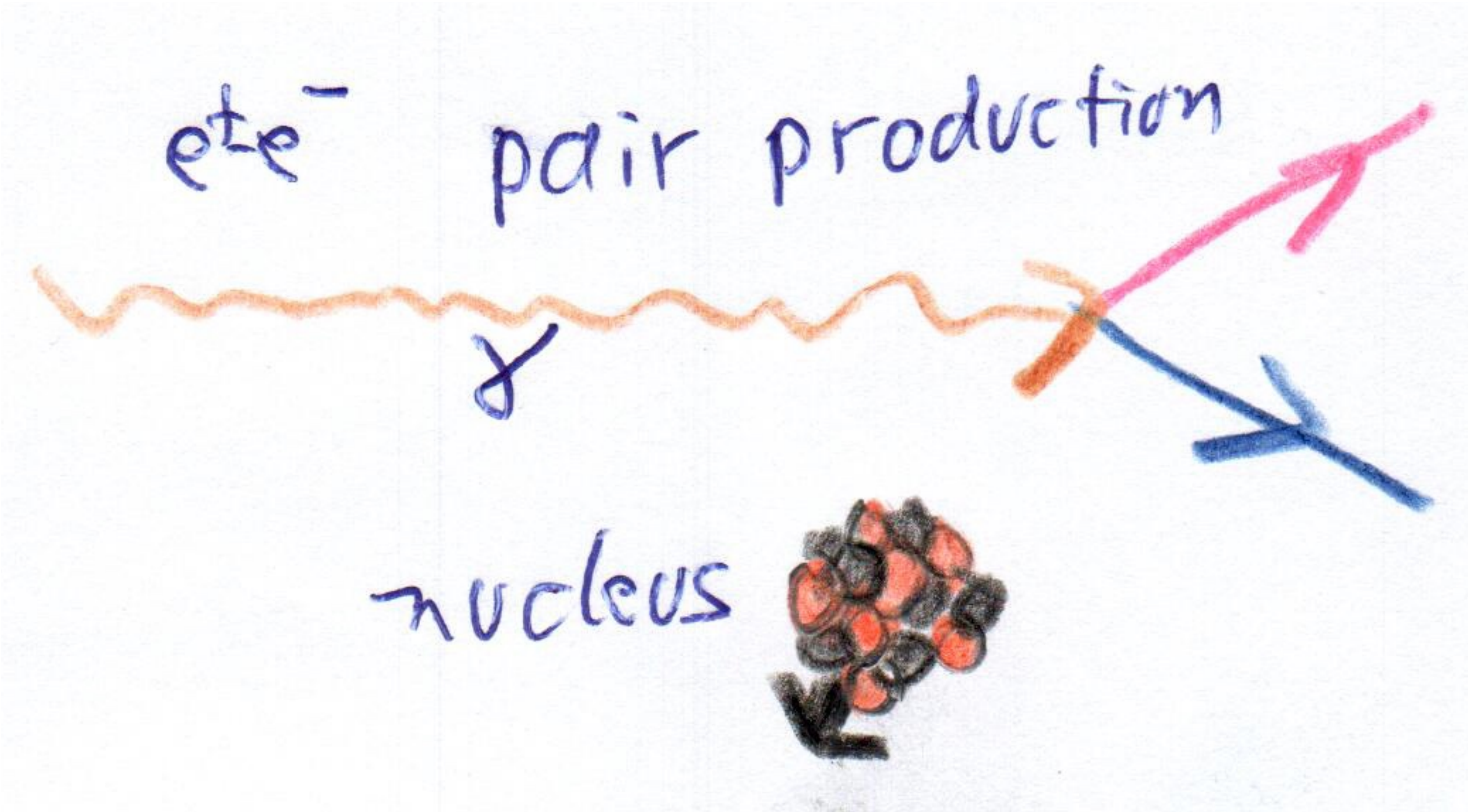


$$h\nu = (2\gamma mc^2)\underbrace{\frac{v}{c}}_{<1}\underbrace{\cos(\theta)}_{\leq 1}$$

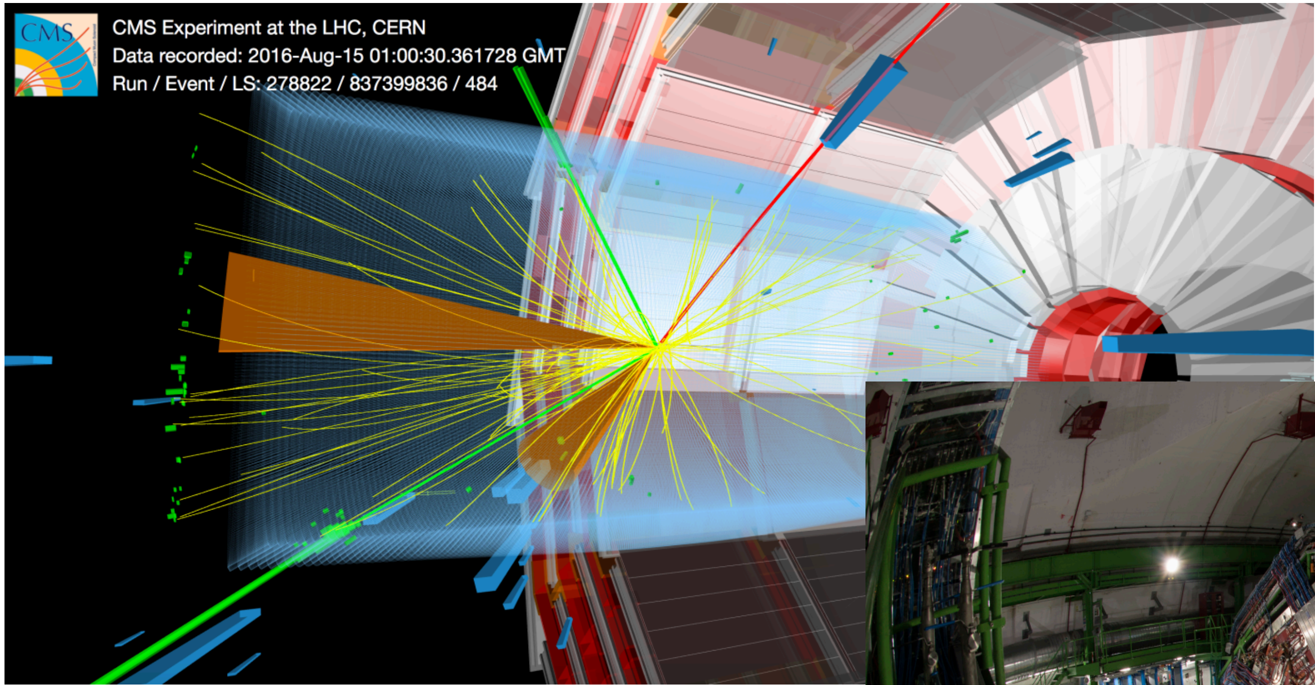
Now cannot fulfill energy and momentum conservation simultanously!!!

Pair production

Requires γ-rays, and the presence of a heavy object (e.g. nucleus) to take up some of the photon's momentum.

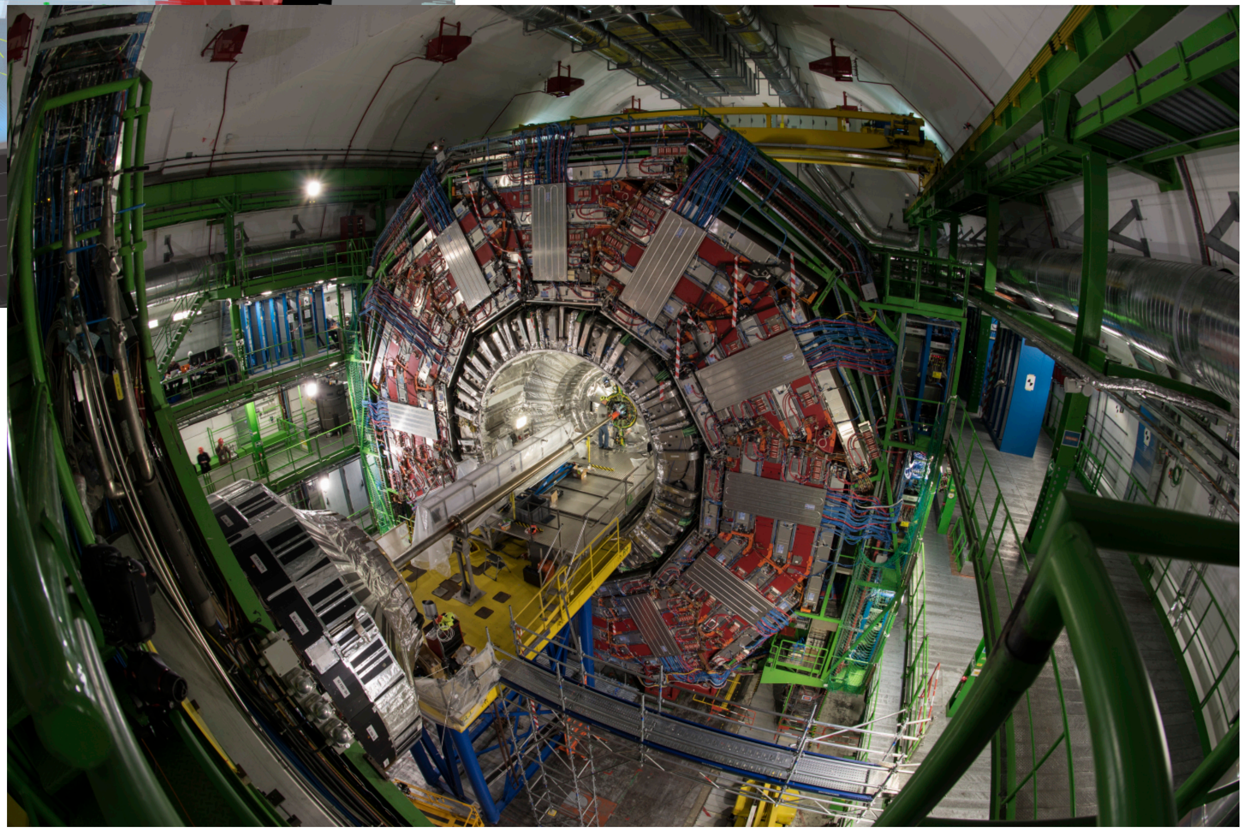


(qq) Pair production galore
LHC detector trace



https://cds.cern.ch/record/2649553

Cern, CMS detector



Light matter interaction

Note: (i) Photo-effect, (ii) Compton scattering, (iii) Pair production all need energetic photons hitting a target.
So **which** happens when we do that?