# Phys106, II-Semester 2019/20, Tutorial 6, [No date, provided for interest and revision]

**Stage 1** Do any items from Tutorial 5 that you did not yet complete last week or at home.

**Stage 2** Discuss on the table what is meant by dispersion. Then explore dispersion for a square-ish wave made of four sine waves using this web applet . Read the page and follow the instructions on it. What happens to the square wave at late times, when you change all four frequencies differently by a small random amount $\sim 20\%$ up or down? *The app above is part of a larger collection of online wave related apps that you might find useful for understanding week 3 or follow up courses.*

**Stage 3 (bonus material)**

(i) If you want to understand also **why** Eq. (43), (44), (46), (47) are valid, it is helpful to review the concept of **vectors** and of a **basis** for a vector space. Let $\mathbf{v} = [v_x, v_y, v_z]^T$ be a 3-component (3D) vector. We can write it in the basis $\mathbf{i} = [1, 0, 0]^T$, $\mathbf{j} = [0, 1, 0]^T$, $\mathbf{k} = [0, 0, 1]^T$ as $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$. But we can choose another basis as well, for example $\mathbf{i}' = (\mathbf{i} + \mathbf{k})/\sqrt{2}$, $\mathbf{j}' = \mathbf{j}$, $\mathbf{k}' = (-\mathbf{i} + \mathbf{k})/\sqrt{2}$. Use the board to draw both bases into a 3D coordinate system.

(ii) We want to now express any vector $\mathbf{v}$ in the new basis as $\mathbf{v} = v'_x\mathbf{i}' + v'_y\mathbf{j}' + v'_z\mathbf{k}'$, where $v'_x = \mathbf{i}' \cdot \mathbf{v}$ etc. Here the symbol $\cdot$ denotes the *scalar product* of two vectors. It is calculated via $\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z$. Test all this using the example vector $\mathbf{v} = [1, 1, 2]$.

(iii) Now consider the following: (a) The above description works for any dimension $d$ of the vector, $\mathbf{v} = [v_1, v_2, \cdots v_d]^T$, i.e. vectors can have 100 components, then the vector space is 100 dimensional. (b) It turns out functions $f(x)$ can be viewed as an $\infty$-dimensional vector. To heuristically understand this consider the figure below: We "discretize space" x by dividing it into lots and lots of points. We can then write the function into a vector $\mathbf{f} = [f(x_1), f(x_2), \cdots, f(x_j), \cdots f(x_J)]^T$. In the end we let $J \to \infty$. (c) It turns out the functions $\varphi_n(x) = \cos(2\pi n x/L)$ for $n = 0, 1, 2, \cdots$ form a **basis** for the vector space of all even functions with period $L$ [proof, see math course]. After we realized this, in analogy to writing *any vector* as $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$, we can write *any even periodic function* as $f(x) = \sum_{n=0}^{\infty} g_n \varphi_n(x)$. The coefficients are $g_n \sim \int f(x)\varphi_n(x)dx$ which, if we discretize space $x$ again, we can understand as formally analogous to a scalar product, since $g_n \sim \int f(x)\varphi_n(x)dx \approx \sum_j f(x_j)\varphi_n(x_j)\Delta x$, see Fig. 1.

(iv) In summary we can thus understand the Fourier series as "expanding the function f(x) in terms of a basis for the space of all functions, provided by the set of cosines with all possible wave lengths".

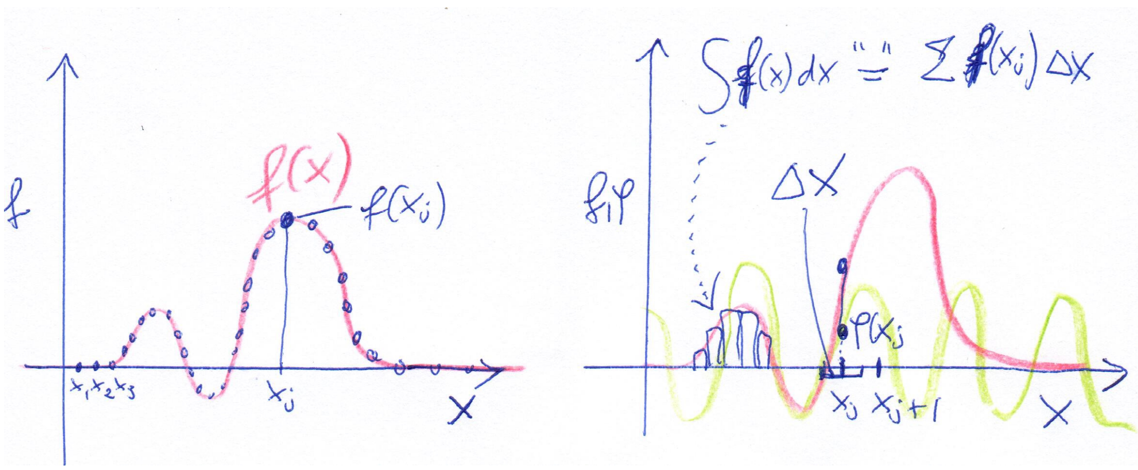


Figure 1: (left) We can view a function as a high-dimensional vector, where each vector element $j$ is the function value $f(x_j)$ at a specific discrete point $x_j$. (right) When viewing the integration $\int dx f(x)\varphi(x)$ in the discretized form, it becomes structurally analogous to a scalar product.