Phys106, II-Semester 2019/20, Assignment 3

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1. Write the Rayleigh Jeans law in terms of wavelength, i.e. we want $u(\lambda)d\lambda$, the total energy within a wavelength interval $d\lambda$. Take care in converting the differential $d\nu$.

   Solution:

   The Rayleigh Jeans law in the terms of frequency is:

   $$u(\nu)d\nu = \frac{8\pi k_b T \nu^2}{c^3} d\nu.$$

   We know the relationship between frequency ν and wavelength λ;

   $$\nu = c/\lambda \Rightarrow \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2} \Rightarrow d\nu = -\frac{c}{\lambda^2} d\lambda.$$

   Using the above two relationship we can write the Rayleigh Jeans law in terms of wavelength

   $$u(\lambda)d\lambda = \frac{8\pi k_b T}{\lambda^4} d\lambda.$$

2. Ultraviolet light of wavelength 200 nm and intensity 3.00 W/m^2 is directed at a Potassium surface. (a) Find the maximum kinetic energy of the photoelectrons (For the workfunction (ϕ) of Potassium see the book "Concept of Modern Physics by Arthur Beiser") (b) if 0.5 percent of the incident photons produce a photoelectron, how many electrons are emitted per second if the surface has an area of 4 cm^2. *Recall 1 Watt = 1J/s, i.e. energy per unit time.*

   Solution:

   (a) *The photon energy is $E = h\nu = 6.199\ eV$. We find in the literature a work-function for Potassium of $\phi = 2.29\ eV$. The kinetic energy of electrons is then:*

   $$K.E. = h\nu - \phi = 3.91eV,$$

   (b) *The energy of a single photon is $E_p = h\nu = 9.93 \times 10^{-19} J$ and the total power (energy per second) falling on the surface of Lithium is,*

   $$W = Intensity \times area = (3.0W/m^2) \times (4 \times 10^{-4} m^2) = 12 \times 10^{-4} J/s.$$

   The number of photon per second falling on the surface of Lithium is then,

   $$n_p = W/E_p \approx 1.2 \times 10^{15}/s.$$

   The number of electrons emitted per second from the surface of Lithium is,

   $$n_e = 0.005 \times n_p = 6.0 \times 10^{12}/s.$$

3. Consider the light emitted by a 35 W room lamp. Assume this is all yellow light with $\lambda = 600$ nm (in reality it is a mix of colors). How many photons per second are there? Why would quantum physics frequently be not that important in determining what happens due to this light? *Solution:*

   The energy of a single photon in yellow light is,

   $$E_p = h\nu = 3.31 \times 10^{-19} J.$$

   Thus the number of photons per second emitted by the lamp is,

   $$n_p = 35W/E_p = 1.0 \times 10^{20}/s.$$

   This is a very large number and thus the fact that the photon number can only take discrete values will be mostly irrelevant. In other words, the difference between 1.0×10^{20} *and* $1.0 \times 10^{20} - 1$ *will typically be negligible.*

4. Derive Eq. (20) of the lecture, that the number of standing waves within a cavity that have a frequency between ν and $\nu + d\nu$ is

   $$G(\nu)d\nu = \frac{8\pi\nu^2}{c^3}d\nu \tag{1}$$

   Proceed as follows:

   (i) Confirm that condition (16) for a standing wave can be re-written in terms of wave number as $k = \frac{\pi}{L}n$ for $n = 1, 2, 3, \cdots$.

   (ii) 3D waves have a wave vector $\mathbf{k} = [k_x, k_y, k_z]^T$. The relation to frequency is $2\pi\nu/|\mathbf{k}| = c$.

   (iii) Now for 3D standing waves that fit into a cubic 3D cavity of volume $V = L^3$, the condition is $k_j = \frac{\pi}{L}n_j$ for $n_j = 1, 2, 3, \cdots$ and $j \in \{x, y, z\}$.

   (iv) That means, that in the space of all vectors $\mathbf{k}$, and allowed standing wave must always lie on one of the dots in Fig. 1. This space is called reciprocal space.

   (v) To infer $G(\nu)d\nu$, we have to now "count" the number of dots within a spherical shell of radius $k(\nu)$ and thickness $dk(d\nu)$. We use that each dot effectively occupies a volume $\left(\frac{\pi}{L}\right)^3$. Also take into account that each wave can have two different polarisations (directions of the E-field vector within the wave), so we have to count it twice.

   (vi) Divide by the total volume of the cavity to get the number of waves *per unit volume.*

   (vii) Use all these steps to show Eq. 1 above.

   Solution: (i): We know that the relationship between the λ *and* k *is,*

   $$k = 2\pi/\lambda.$$

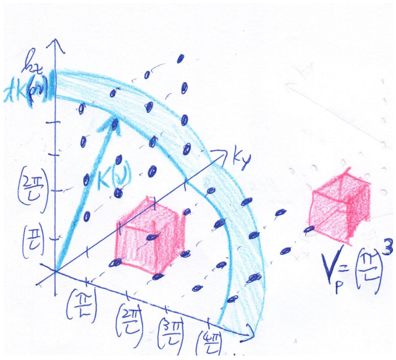

Figure 1: Sketch of reciprocal space. Big dots indicate allowed 3D wave-vectors. Red shade indicates the volume effectively taken by one of these dots.

Putting this into eq. 16 we get,

$$k = \frac{\pi}{L} n.$$

In 3D the wave vector have the form,

$$\mathbf{k} = [k_x, k_y, k_z]^T = k_x \hat{x} + k_y \hat{y} + k_z \hat{z},$$

where, $k_x = \frac{\pi}{L} n_x, k_y = \frac{\pi}{L} n_y$ *and* $k_z = \frac{\pi}{L} n_z$ *are the allowed wave vectors in the direction of x, y and z respectively such that they are associated with one of the points of the lattice in k-space (see Fig.1). Above* $\hat{x}$, $\hat{y}$ *and* $\hat{z}$ *are unit vectors in the x,y,z directions.*

We can see from the figure that each point in k-space effectively occupies a volume $V_p = \left(\frac{\pi}{L}\right)^3$.

(v): The volume of a spherical shell of radius $k(\nu)$ *and thickness* $dk(\nu)$ *is* $4\pi k^2 dk$ *but we are only interested in the positive value of* n_x, n_y *and* n_z *so we need to only consider the one-eighth of volume of spherical shell i.e.* $V_s = \frac{1}{8} \times 4\pi k^2 dk$.

Now the number of standing waves in the shell is the number of points times two (polarisation)

$$N(k)dk = 2 \times V_s / V_p = \frac{k^2 L^3 dk}{\pi^2}.$$

Here the factor 2 takes care of polarisation of standing waves.

(vi): To get the number of standing waves per unit volume, we divide by the total volume of the cavity $V_{cav} = L^3$*:*

$$G(k)dk = \frac{k^2 dk}{\pi^2},$$

Now we can use the relationship $2\pi\nu / |\mathbf{k}| = c$ *to write this in term of frequency as,*

$$G(\nu)d\nu = \frac{8\nu^2 d\nu}{c^2} d\nu.$$

where we have used a similar conversion for the differentials as in the first question.