Phys106, II-Semester 2019/20, Assignment 4 solution

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1. (a) Since photons carry momentum, when they reflect off a mirror they exert a force on that mirror, called radiation pressure. From your knowledge of the photon momentum, infer a simple equation how radiation pressure depends on light intensity. Use momentum conservation laws for that, see Fig. 1.

   Hint: You may also need the following relations: $F = dp/dt$ *(Force is change of momentum per unit time),* $P = F/A$ *(Pressure is force per unit Area). Recall assignment 3 where you had already looked at the relation between light intensity and photons contained in it.*
   (b) Solar radiation has an intensity of about $I_{sol} = 1400\ W/m^2$ at the earth orbit.

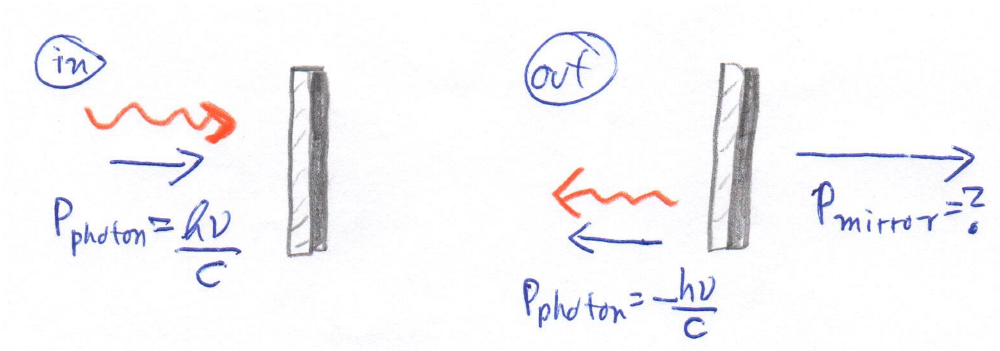

Figure 1: Momentum conservation diagram when a photon reflects off a mirror.

   If you build a square solar sail (very thin mirror type sheet) of area 1 km^2 and tie it to a space-probe of mass $m = 1$ kg, what acceleration does the space-probe experience? Is this a great way to build spaceships?
   Solution:
   (a) For N photons falling on the mirror:
   Initial momentum of photons $= \frac{Nh\nu}{c}$ *Final momentum of photons* $= -\frac{Nh\nu}{c}$
   Initial momentum of mirror $= 0$, *Final momentum of mirror* $= p_{mirror}$
   Therefore according to the momentum conservation

$$\frac{Nh\nu}{c} + 0 = -\frac{Nh\nu}{c} + p_{mirror}$$

$$p_{mirror} = \frac{2Nh\nu}{c}$$

   Radiation pressure P_{rad} *is given by*

$$P_{rad} = \frac{F_{mirror}}{A}$$

where $F_{mirror} = \frac{dp_{mirror}}{dt}$ is the force exerted by the photons on the mirror and A is the area of the mirror. Suppose that in a time interval t N photons have indeed reflected off the mirror, then the momentum has changed by p_{mirror} derived above, thus we can write

$$P_{rad} = \frac{F_{mirror}}{A} = \frac{1}{A}\frac{dp_{mirror}}{dt} = \frac{2Nh\nu}{ctA} \tag{1}$$

Intensity of light has units of (energy per unit area A per unit time t) and is thus given by

$$I = \frac{Nh\nu}{At}$$

Hence we can write

$$P_{rad} = \frac{1}{A} \times \frac{d}{dt}\left(\frac{2It}{c}\right) \implies \boxed{P_{rad} = \frac{2I}{c}}$$

(b) So the force due to the solar radiation on the mirror is given by

$$F_{mirror} = P_{rad}A$$

Therefore the acceleration experienced by the space-probe is

$$a = \frac{F_{mirror}}{m} = \frac{2IA}{mc} \implies \boxed{a = 9.34\ m/s^2}$$

This is almost the same as the acceleration we feel due to earth gravity standing on its surface. However 1 km^2 is a huge sail, and 1 kg a small probe. Further out from the star, acceleration will be less and less. Thus a solar sail is somewhat of a limited plan. The advantage is, that it can accelerate a probe for a very long time.

2. Show that electron positron pair production cannot occur in free space without looking at the lecture calculation. Only start from the diagram on page three of section 2.2.7), then apply energy and momentum conservation with the relativistic formulae given in section 2.2.5).
   Solution: For solution you may look at the lecture note. Please find here a slightly different version of proof. Looking at the diagram given on page three of 2.2.7 we can write energy conservation for the process

$$E_\gamma = E_{e^-} + E_{e^+}$$

where photon energy $E_\gamma = p_\gamma c$, scattered electron energy $E_{e^-}^2 = p_e^2c^2 + m_e^2c^4$ and scattered positron energy $E_{e^+}^2 = p_p^2c^2 + m_e^2c^4$

Therefore for conservation of energy

$$p_\gamma c = \sqrt{(p_e^2 c^2 + m_e^2 c^4)} + \sqrt{p_p^2 c^2 + m_e^2 c^4}$$

and we know that $p_p = p_e$ *from conservation of momentum perpendicular to the original photon direction. So*

$$p_\gamma = 2\sqrt{(p_e^2 + m_e^2 c^2)}$$

Now conserving linear momentum in the original direction of photon

$$p_\gamma = p_e cos(\theta) + p_p cos(\theta) = 2p_e cos(\theta)$$

Equating two equations we get

$$p_e cos(\theta) = \sqrt{(p_e^2 + m_e^2 c^2)}$$

$$cos(\theta) = \sqrt{1 + \frac{m_e^2 c^2}{p_e^2}}$$

As $cos(\theta)$ *cannot exceed 1 we see that this is impossible. Hence elctron positron pair production cannot occur.*

3. Calculate the Compton wavelength of (i) a tau neutrino (let us assume a rest mass 0.05 eV/c^2. The true value is still not known) (ii) a muon, (iii) a proton (iv) a Uranium+ atom ? Which of these will show the nicest signal in X-Ray scattering? What is the maximum wavelength shift in scattering from protons?
Solution: Compton wavelength is given by

$$\lambda_c = \frac{h}{mc}$$

(i) λ_c *for tau neutrino*$= 2.48 \times 10^{-5}\ m$
(ii) λ_c *for muon (mass*$= 105.658\ MeV/c^2$*)* $= 1.173 \times 10^{-14}\ m$
(iii) λ_c *for proton* $= 1.321 \times 10^{-15}\ m$
(iv) λ_c *for Uranium+ atom (*$\sim {}^{235}U$ *with mass* $235 \times 931.494 MeV/c^2$*)* $= 5.664 \times 10^{-18}\ m$

The wavelength range of X-rays is about 10^{-8} *to* 10^{-12} *m. So, none of the above will give a nice X-Ray scattering signal, since they all lie outside this range. [Exercise: try this for electron]*
Maximum wavelength shift in scattering from a proton is for direct back scattering, at $\phi = \pi$, *where* $\cos(\phi) = -1$. *It is thus given by*

$$\lambda' - \lambda = 2\lambda_c = 2 \times 1.324 \times 10^{-15} m = 2.64 \times 10^{-15}\ m.$$