Phys106, II-Semester 2019/20, Assignment 2 solution

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1. A function f is called periodic with period T if $f(t+T) = f(t)$. The time average of a such a function is given by

$$\overline{f(t)} = \frac{1}{T}\int_0^T f(t)dt. \tag{1}$$

Calculate $\overline{\cos(\omega t)}$, $\overline{\sin(\omega t)}$, $\overline{\cos^2(\omega t)}$, $\overline{\sin^2(\omega t)}$. What average would you get when averaging over a larger number of periods nT?

Solution: We do both questions together, for the first one set $n = 1$. For a large number of periods nT:

$$\overline{\cos(\omega t)} = \frac{1}{nT}\int_0^{nT}\cos(\omega t)dt = \frac{1}{nT}\left[\frac{1}{\omega}\sin(\omega t)\right]_0^{nT} = \frac{1}{nT}\left(\frac{\sin(\omega nT) - \sin(0)}{\omega}\right)$$
$$= \frac{\sin(nT\omega)}{(nT\omega)} \overset{T\omega=2\pi}{=} \frac{\sin(n2\pi)}{(n2\pi)} = 0, \tag{2}$$

$$\overline{\sin(\omega t)} = \frac{1}{nT}\int_0^{nT}\sin(\omega t)dt = -\frac{1}{nT}\left(\frac{\cos(\omega nT) - \cos(0)}{\omega}\right) = \frac{1-\cos(n2\pi)}{(n2\pi)} = 0, \tag{3}$$

$$\overline{\cos^2(\omega t)} = \frac{1}{nT}\int_0^{nT}\frac{1+\cos(\omega t)}{2}dt \overset{Eq. (2)}{=} \frac{1}{nT}\int_0^{nT}\frac{1}{2}dt = \frac{nT}{2nT} = \frac{1}{2}, \tag{4}$$

$$\overline{\sin^2(\omega t)} = \frac{1}{nT}\int_0^{nT}\frac{1-\cos(\omega t)}{2}dt \overset{Eq. (3)}{=} \frac{1}{nT}\int_0^{nT}\frac{1}{2}dt = \frac{nT}{2nT} = \frac{1}{2}. \tag{5}$$

2. Do the missing steps in the lecture for the derivation of the double slit interference pattern:

   (i) Start with Fig. 2 in week 3, using geometry, express r_1 and r_2 through z, L, and the angle θ of the vectors connecting the slits and location z on the screen. These angles are slightly different, make the approximation that they are equal, as shown in the attached sketch Fig. 1.

   (ii) Also approximate the $r_{1,2}$ dependence of the prefactors of y_1 and y_2 (Fig. 2) as equal $r_1 \approx r_2 \approx L$. Do not do this approximation for $r_{1,2}$ within the argument of the wave (sin) in Fig. 2.

   (iii) Within $y(z,t)$, split the space and time dependence in the trigonometric functions into the form $f(t)g(x)$ using the trigonometric identity $\sin a + \sin b = 2\sin(\frac{a+b}{2})\cos(\frac{a-b}{2})$.

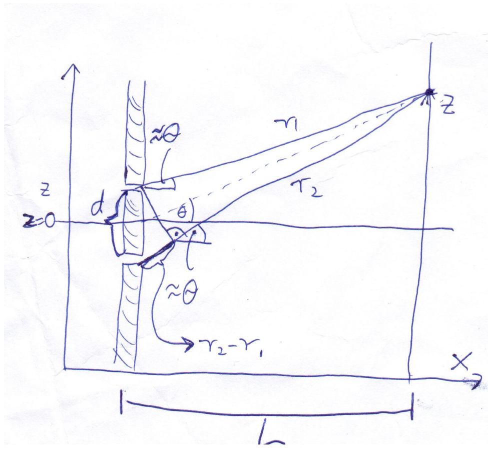

Figure 1: Sketch of variables for double slit geometry.

(iv) Now calculate the intensity $I(z,t) = |y(z,t)|^2$, and perform the long time average over this using your results from question one. Simply define here $I_0 = 2A^2/L^2$. Compare with the lecture or book.

Solution: (i)

See figure (1a),
path difference, $(r_2 - r_1) = d\sin\theta$ *and*
$r_1 = \frac{L}{\cos\theta}$

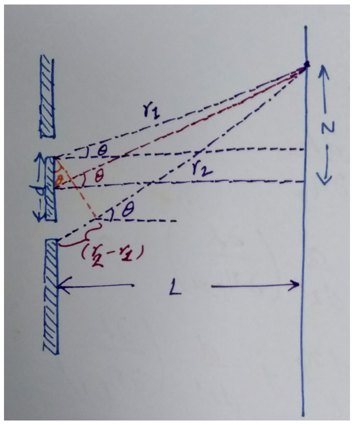

Fig.(1a)

(ii) Under the approximation $r_1 \approx r_2 \approx L$,

$$y_1(r_1,t) = \frac{A}{L}\sin(kr_1 - \omega t)$$

$$y_2(r_2,t) = \frac{A}{L}\sin(kr_2 - \omega t)$$

(iii) Now we add the two waves to find

$$\begin{aligned} y(z,t) &= y_1(r_1(z),t) + y_2(r_2(z),t) \\ &= \frac{A}{L}[\sin(kr_1 - \omega t) + \sin(kr_2 - \omega t)] \end{aligned}$$

$$= \frac{2A}{L}\sin\left(\frac{k(r_2+r_1)-2\omega t}{2}\right)\cos\left(\frac{k(r_2-r_1)}{2}\right)$$
$$= \frac{2A}{L}\sin(\phi-\omega t)\cos\left(\frac{k(r_2-r_1)}{2}\right)$$

where $\phi = \frac{k(r_1+r_2)}{2} = \frac{\phi_1+\phi_2}{2}$ *is the phase of the resultant wave.*

(iv) For the intensity we square the wave (in reality there may be other factors, but a square of wave amplitude will be there):

$$|y(z,t)|^2 = \frac{4A^2}{L^2}\sin^2(\phi-\omega t)\cos^2\left(\frac{k(r_2-r_1)}{2}\right)$$

so the averaged intensity is

$$I(z,t) = \overline{|y(z,t)|^2} \overset{question1}{=} \frac{4A^2}{L^2}\left\{\frac{1}{2}\right\}\cos^2\left(\frac{k(r_2-r_1)}{2}\right)$$

Now using $I_0 = \frac{2A^2}{L^2}$*, we find*

$$I(z,t) = I_0\cos^2\left(\frac{k(r_2-r_1)}{2}\right).$$

From (i), $(r_2-r_1) = d\sin\theta$ *hence*

$$I(z,t) = I_0\cos^2\left(\pi d\frac{sin\theta}{\lambda}\right),$$

as in the lecture.

3. The following wave equation is called "sine-Gordon equation":

$$\frac{\partial^2}{\partial t^2}f(x,t) - \frac{\partial^2}{\partial x^2}f(x,t) + \sin\left[f(x,t)\right] = 0. \tag{6}$$

Determine if the superposition principle holds for this equation and why or why not.
Solution:

The given equation is a non-linear differential equation in $f(x,t)$ *and superposition principle does not hold for non-linear differential equations.*

For example let $f_1(x,t)$ *and* $f_2(x,t)$ *are two solutions of given differential equation then*

$$\frac{\partial^2}{\partial t^2}f_1(x,t) - \frac{\partial^2}{\partial x^2}f_1(x,t) + \sin\left[f_1(x,t)\right] = 0$$
$$\frac{\partial^2}{\partial t^2}f_2(x,t) - \frac{\partial^2}{\partial x^2}f_2(x,t) + \sin\left[f_2(x,t)\right] = 0$$

If superposition principle hold then we would have,

$$\frac{\partial^2}{\partial t^2}(f_1(x,t)+f_2(x,t)) - \frac{\partial^2}{\partial x^2}(f_1(x,t)+f_2(x,t)) + \sin\left[(f_1(x,t)+f_2(x,t))\right] = 0$$

which is in general not true because $\sin\left[(f_1(x,t)+f_2(x,t))\right] \neq \sin\left[(f_1(x,t)\right] + \sin\left[f_2(x,t))\right]$.

4. In the lecture we saw that two counter-propagating waves form a standing wave. Show that if one of them has a phase shift φ, as in:

$$y(x,t) = A\sin(kx - \omega t) + A\sin(-kx - \omega t + \varphi) \tag{7}$$

we still get a standing wave, albeit with shifted positions of the minima.
Solution:

$$\begin{aligned}
y(z,t) &= A\sin(kx-\omega t) + A\sin(-kx-\omega t+\varphi) \\
&= \frac{A}{2\pi i}\Big[\exp[i(kx-\omega t)] - \exp[-i(kx-\omega t)] + \exp[i(-kx-\omega t+\varphi)] \\
&\quad - \exp[-i(-kx-\omega t+\varphi)]\Big] \\
&= \frac{A}{2\pi i}\Big[e^{-i\omega t}\Big(e^{ikx} + e^{i(-kx+\varphi)}\Big) - e^{i\omega t}\Big(e^{-ikx} + e^{i(kx-\varphi)}\Big)\Big] \\
&= \frac{A}{2\pi i}\Big[e^{-i(\omega t-\frac{\varphi}{2})}\Big(e^{i(kx-\frac{\varphi}{2})} + e^{-i(kx-\frac{\varphi}{2})}\Big) - e^{i(\omega t-\frac{\varphi}{2})}\Big(e^{-i(kx-\frac{\varphi}{2})} + e^{i(kx-\frac{\varphi}{2})}\Big)\Big] \\
&= \frac{2A}{2\pi i}\Big[e^{-i(\omega t-\frac{\varphi}{2})}\cos(kx-\frac{\varphi}{2}) - e^{i(\omega t-\frac{\varphi}{2})}\cos(kx-\frac{\varphi}{2})\Big] \\
&= \frac{2A}{2\pi i}\Big[e^{-i(\omega t-\frac{\varphi}{2})} - e^{i(\omega t-\frac{\varphi}{2})}\Big]\cos(kx-\frac{\varphi}{2}) \\
&= -2A\sin(\omega t-\frac{\varphi}{2})\cos(kx-\frac{\varphi}{2})
\end{aligned}$$