Phys106, II-Semester 2019/20, Assignment 1 solution

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1. Show that $x(t) = Ae^{-\frac{\gamma}{2}t}\cos(\tilde{\omega}t + \varphi)$ is a solution of the differential equation for the damped harmonic oscillator without driving (Eq. (1) of lecture with $F_0 = 0$). Here $\tilde{\omega} = \sqrt{\omega_0^2 - \gamma^2/4}$. What does the solution imply? Discuss its physical meaning and that of all parameters in it.
   <u>Solution</u>: *Equation for a damped harmonic oscillator:*

$$\frac{d^2x(t)}{dt^2} = -\omega_0^2 x(t) - \gamma\frac{dx(t)}{dt} \tag{1}$$

To check if $x(t) = Ae^{-\frac{\gamma}{2}t}\cos(\tilde{\omega}t + \varphi)$ is a solution to above equation, we will substitute this $x(t)$ in the above equation. A true solution would give LHS = RHS. First, we calculate the derivatives (applying chain rule):

$$\begin{aligned}\frac{dx(t)}{dt} &= -\frac{A\gamma}{2}e^{-\gamma t/2}\cos(\tilde{\omega}t + \varphi) - A\tilde{\omega}e^{-\frac{\gamma}{2}t}\sin(\tilde{\omega}t + \varphi) \qquad (2)\\ \frac{d^2x(t)}{dt^2} &= \frac{A\gamma^2}{4}e^{-\gamma t/2}\cos(\tilde{\omega}t + \varphi) - A\tilde{\omega}^2 e^{-\gamma t/2}\cos(\tilde{\omega}t + \varphi) + A\gamma\tilde{\omega}e^{-\frac{\gamma}{2}t}\sin(\tilde{\omega}t + \varphi)\end{aligned}$$

Substituting $x(t)$ and $dx(t)/dt$ in RHS of Eq. (1):

$$\begin{aligned}-\omega_0^2 x(t) - \gamma\frac{dx(t)}{dt} &= -\tilde{\omega}^2 x(t) - \frac{\gamma^2}{4}x(t) - \gamma\frac{dx(t)}{dt}\\ &= -A\tilde{\omega}^2 e^{-\frac{\gamma}{2}t}\cos(\tilde{\omega}t + \varphi) - A\frac{\gamma^2}{4}e^{-\frac{\gamma}{2}t}\cos(\tilde{\omega}t + \varphi)\\ &\quad + \frac{A\gamma^2}{2}e^{-\gamma t/2}\cos(\tilde{\omega}t + \varphi) + A\gamma\tilde{\omega}e^{-\frac{\gamma}{2}t}\sin(\tilde{\omega}t + \varphi)\\ &= \frac{d^2x(t)}{dt^2} = LHS \qquad (3)\end{aligned}$$

This implies, the motion of a damped harmonic oscillator is still oscillatory, however (a) the amplitude of oscillation acquires time dependence: it decays exponentially; and (b) the frequency oscillation acquires a dependence on damping parameter γ. Physically, it means there are three cases possible:

- $\omega_0^2 - \gamma^2/4 < 0$: *"Overdamped" - motion is not oscillatory*
- $\omega_0^2 - \gamma^2/4 = 0$: *"critical damping" - not oscillatory and rapidly decaying*
- $\omega_0^2 - \gamma^2/4 = 0$: *"underdamped" - oscillatory but decays over time*

2. Do the missing steps in the lecture for the solution of the driven, damped harmonic oscillator. To this end:

   (i) Insert the Ansatz $x(t) = A\sin(\omega t) + B\cos(\omega t)$ into Eq. (1).

(ii) Separately equate coefficients of sin and of cos on the left-hand-side (lhs) and rhs.

(iii) This gives you two equations for the two coefficients A and B. Solve these to get A and B.

(iv) Find a rule how you can express $A\sin(\omega t) + B\cos(\omega t) = C\sin(\omega t + \varphi)$ online or in your math books/lextures, to find C and φ in terms of A and B.

(v) Verify your C and φ agree with those given in the lecture, if need by by plotting them on top of each other.

<u>Solution</u>: *(i) Substitution:*

$$\begin{aligned} -A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t) &= -\omega_0^2(A\sin(\omega t) + B\cos(\omega t)) + F_0\sin(\omega t) \\ &\quad -\gamma(A\omega\cos(\omega t) - B\omega\sin(\omega t)) \\ &= (-A\omega_0^2 + F_0 + B\gamma\omega)\sin(\omega t) + (-B\omega_0^2 - A\gamma\omega)\cos(\omega t) \end{aligned}$$

(ii) Comparing coefficients:

$$-A\omega^2 = -A\omega_0^2 + F_0 + B\gamma\omega; \tag{4}$$

$$-B\omega^2 = -B\omega_0^2 - A\gamma\omega \tag{5}$$

(iii) Solving for A and B:

$$A = \frac{F_0}{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}(\omega_0^2 - \omega^2) \tag{6}$$

$$B = -\frac{\gamma\omega F_0}{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2} \tag{7}$$

(iv) Using:

$$\begin{aligned} A\sin(\omega t) + B\cos(\omega t) &= \sqrt{A^2 + B^2}\Big(\cos\varphi\sin(\omega t) + \sin\varphi\cos(\omega t)\Big) \\ &= \sqrt{A^2 + B^2}\sin(\omega t + \varphi) \end{aligned} \tag{8}$$

where, $\cos\varphi \equiv A/\sqrt{A^2 + B^2}$. *Using this formula, we obtain:*

$$\begin{aligned} C &= \sqrt{A^2 + B^2} \\ &= \sqrt{\frac{F_0^2(\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2)}{(\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2)^2}} \\ &= \frac{|F_0|}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \end{aligned} \tag{9}$$

Calculating φ:

$$\begin{aligned} & \cos\varphi \equiv A/\sqrt{A^2 + B^2} \\ \Longrightarrow\quad & \tan\varphi = \frac{B}{A} = -\frac{\gamma\omega}{\omega_0^2 - \omega^2} \\ \Longrightarrow\quad & \varphi = \arctan\left(\frac{\gamma\omega}{\omega^2 - \omega_0^2}\right) + n\pi\ ; \quad n \in \mathbb{Z} \end{aligned} \tag{10}$$

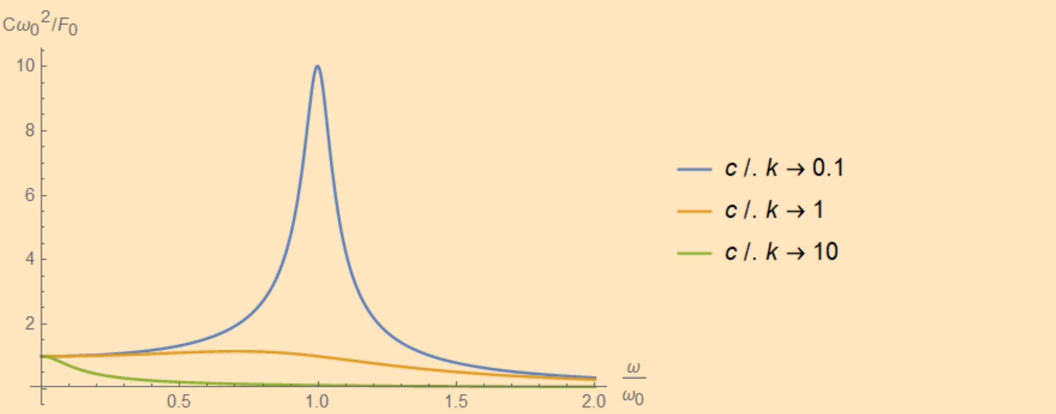
(v) The solution for C matches the one in lecture. The solution for φ matches upto an additive constant $n\pi$ where $n \in \mathbb{Z}$.

3. Use a computer tool to plot the amplitude C and phase angle φ for different parameters. Try to explore some parameters where the curves plotted are much different from those in the lecture, and discuss why.
Solution: Plotting using Mathematica.
To plot C vs. ω/ω_0, we redefine C using a new variable $z \equiv \omega/\omega_0$:

$$C = \frac{|F|_0}{\sqrt{\left(\frac{\gamma}{\omega_0}\right)^2 z^2 + (z^2-1)^2}}$$

We plot the graph for three cases (here, $\gamma/\omega_0 \equiv k$): (a) $k = 0.1$, (b) $k = 1$ and (c) $k = 10$ with a constant F_0 (say $F_0 = 1$):

```
Plot[{c /. k → 0.1, c /. k → 1, c /. k → 10}, {z, 0, 2}, AxesLabel → {ω / ω0, "Cω0^2 / F0"},
 PlotLegends → "Expressions", PlotRange → Full]
```



Cases (b) and (c) give different plots. They correspond to high damping scenario. A generic plot for arctan is given below:

```
Plot[{ArcTan[x], ArcTan[x] + Pi}, {x, -3, 3}, AxesLabel → {x, φ}, PlotLegends → "Expressions"]
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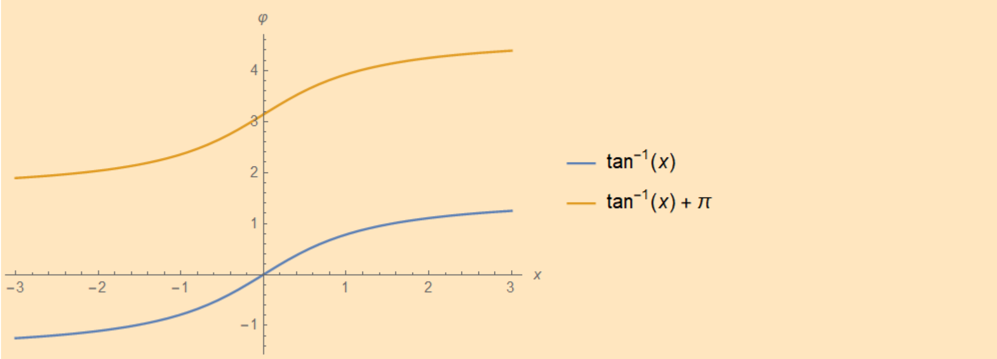


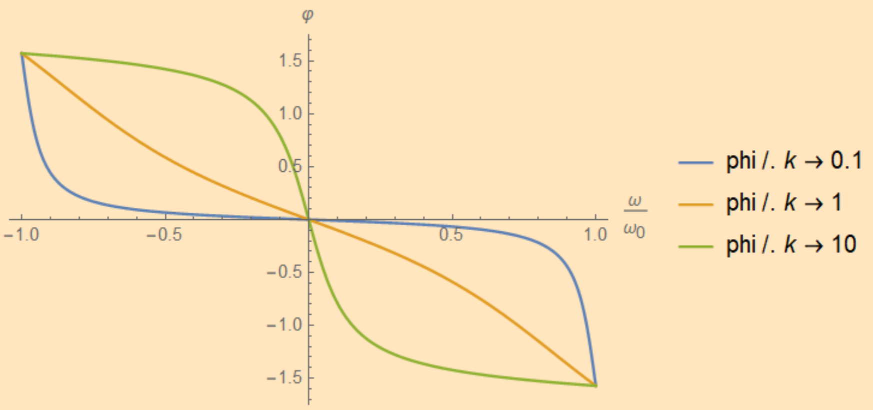
We would like to plot φ vs. ω/ω_0, so we redefine φ using a new variable $u \equiv \omega/\omega_0$

and take $n = 0$ for convenience:

$$\varphi = \arctan\left(\frac{\gamma u}{\omega_0(u^2 - 1)}\right)$$

Like before, plotting three different cases (with, $\gamma/\omega_0 \equiv k$): (a) $k = 0.1$, (b) $k = 1$ and (c) $k = 10$:

```
Plot[{phi /. k → 0.1, phi /. k → 1, phi /. k → 10}, {u, -1, 1}, AxesLabel → {ω / ω0, φ},
 PlotLegends → "Expressions"]
```



In cases (a) and (b), the graph is different from the one in lecture (which matches case (c)).