Week 5

PHY 106 Quantum Physics
Instructor: Sebastian Wüster, IISER Bhopal, 2020

Book: Wave “groups”

ss above only.
ct me if you spot a mistake.

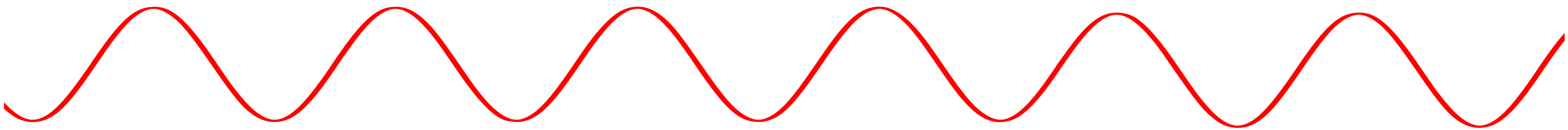
2.3) Wave packets and dispersion

Movies: Elm waves are **also particles**

What if traditional particles (electrons) are also waves?

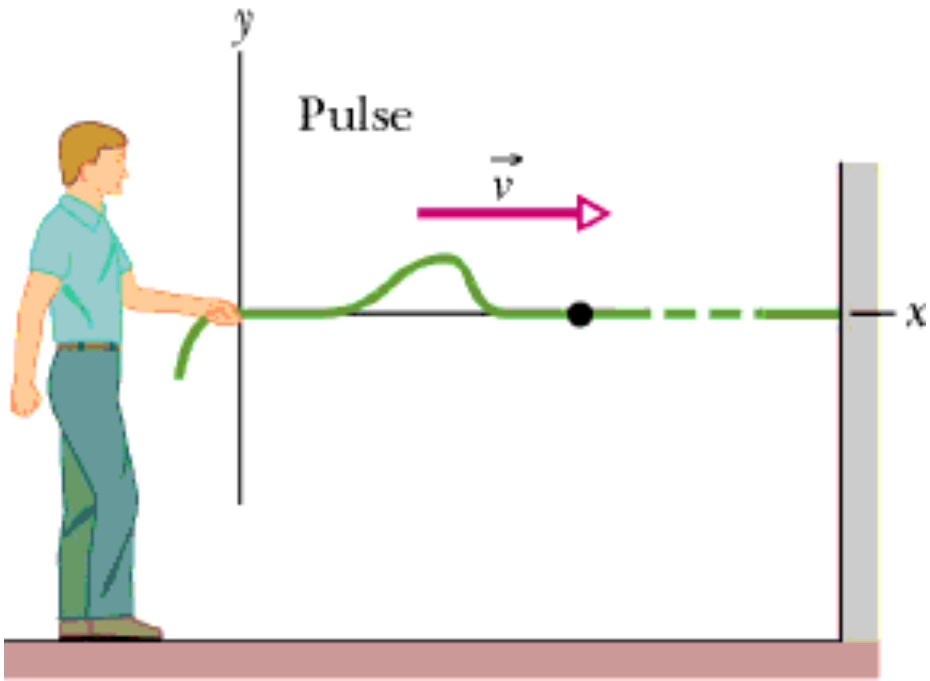
But particle can be **in a specific place**

Extended travelling wave? **Is not!!!**



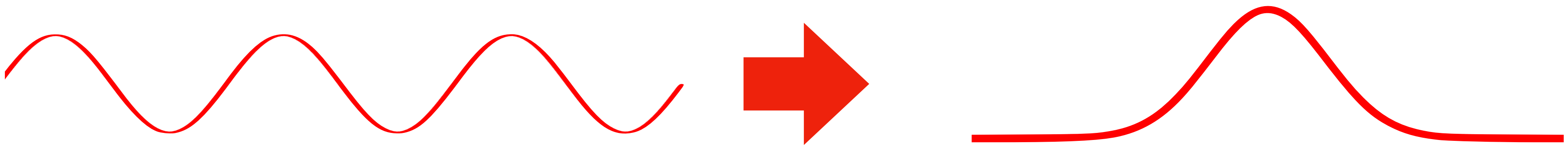
https://phet.colorado.edu/sims/html/wave-on-a-string/latest/wave-on-a-string_en.html

But in our rope app/experiment, can also see wave **pulse**:



Here we could say wave is
in a specific place

How to get from sin/cos wave to pulse?



It is useful to keep discussing (also pulse) waves in terms of *sin* and *cos*, since all our week 3 material will apply!

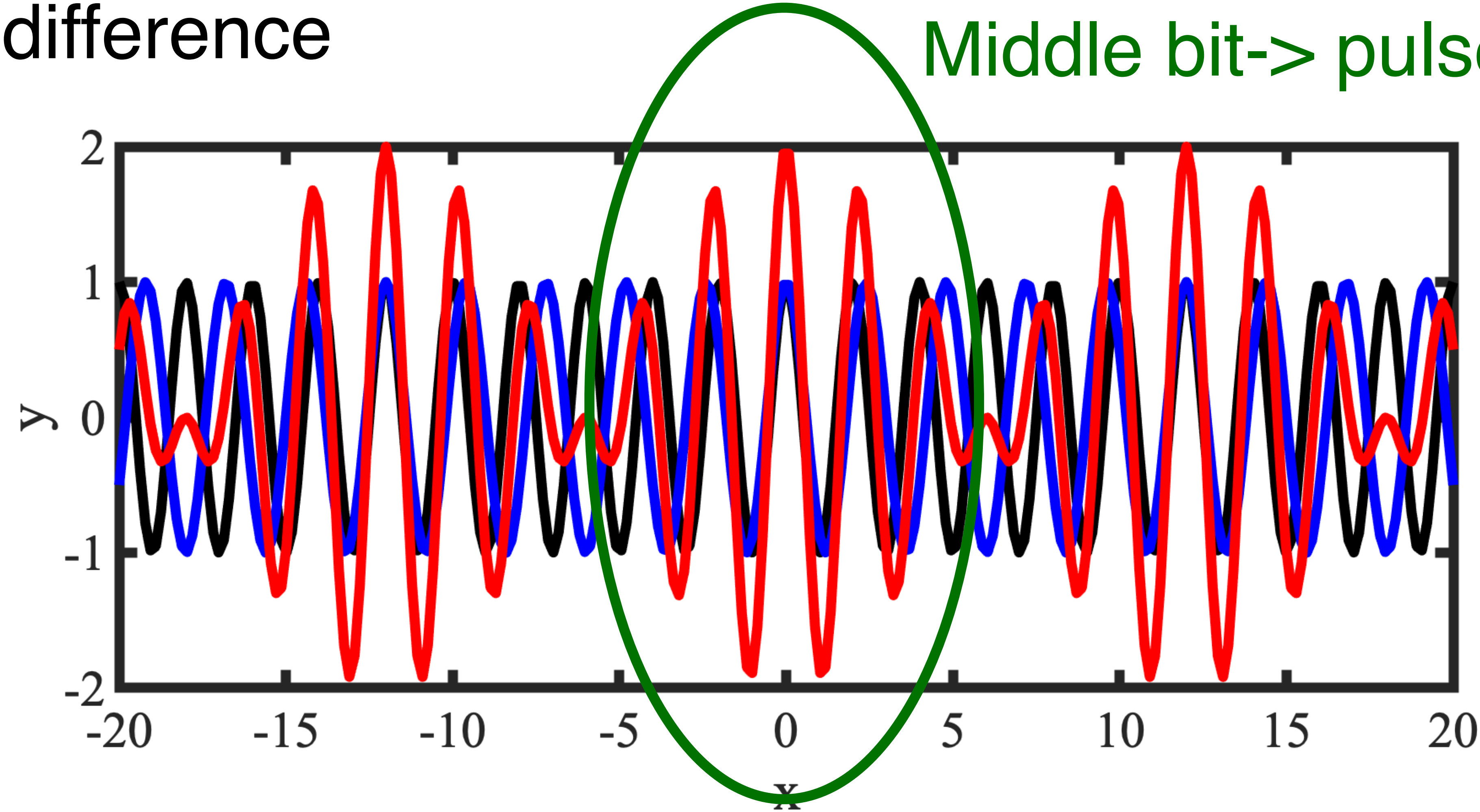
Idea: use superposition principle to combine *different* sine waves into hump?

2.3.1.) Beating of two waves

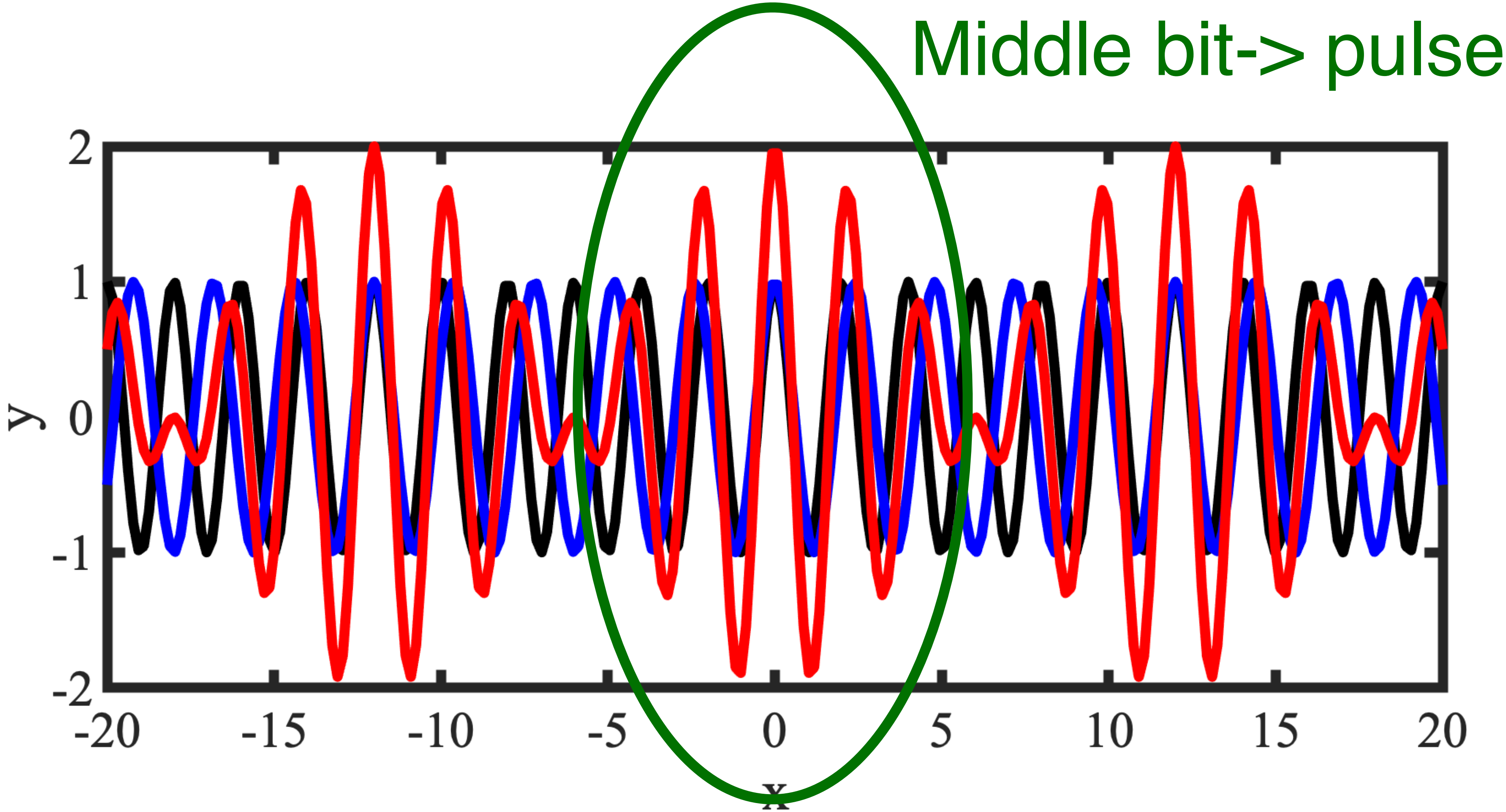
Recall question from tutorial 2:

Adding two sines with slight wavelength difference

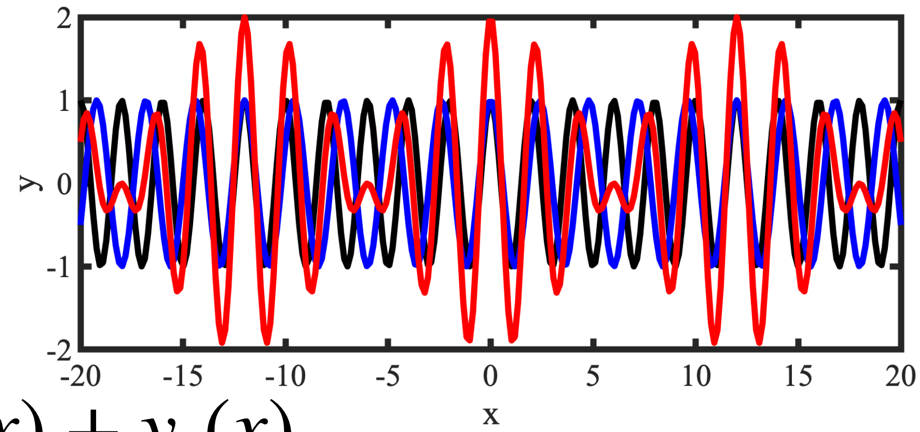
Middle bit-> pulse



Beating of two waves



Beating of two waves



Let's do the math:

$$y_{tot}(x) = y_1(x) + y_2(x)$$

$$y_1(x) = A\cos\left(\frac{2\pi}{\lambda_1}x\right) \quad + \quad y_2(x) = A\cos\left(\frac{2\pi}{\lambda_2}x\right)$$

Use

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$y_{tot}(x) = 2A\cos\left[\left(\frac{\pi}{\lambda_1} + \frac{\pi}{\lambda_2}\right)x\right]\cos\left[\left(\frac{\pi}{\lambda_1} - \frac{\pi}{\lambda_2}\right)x\right]$$

Beating of two waves

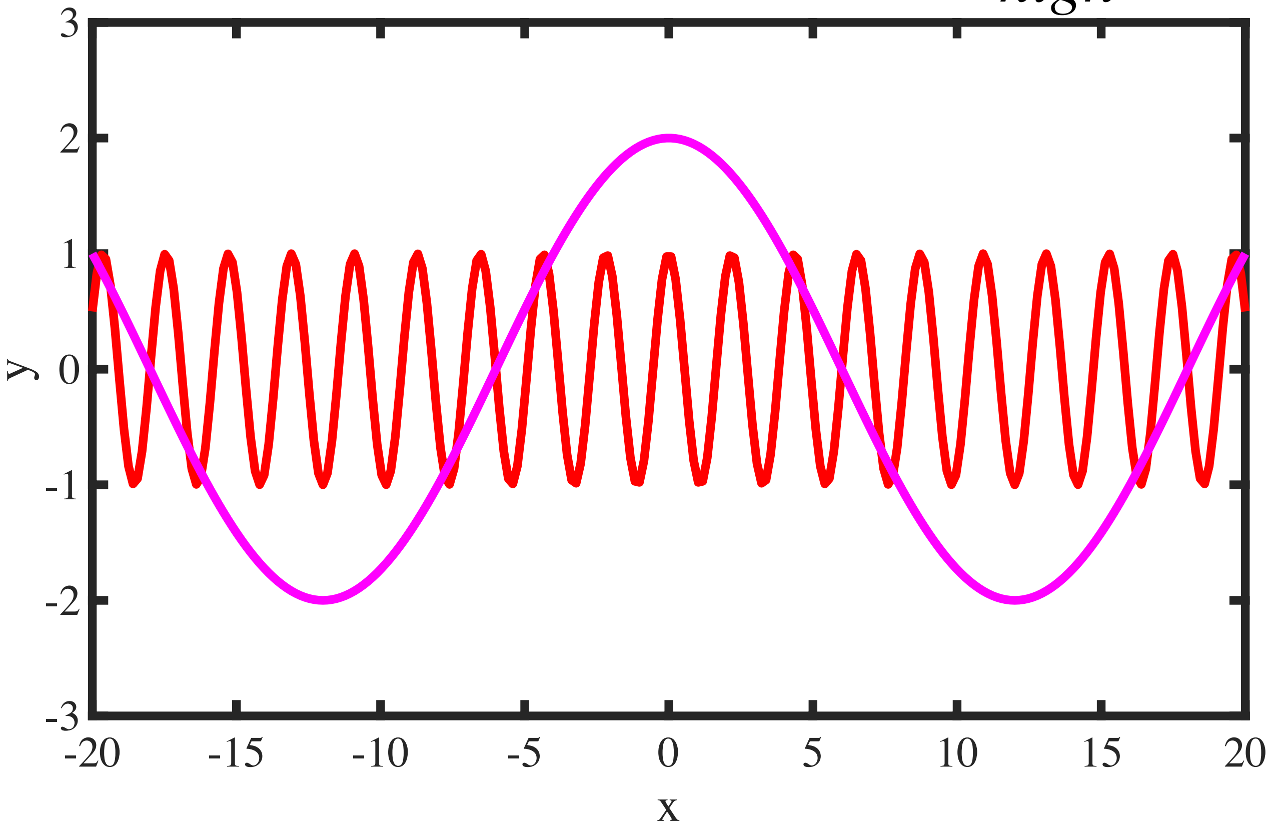
$$y_{tot}(x) = 2A\cos\left[\left(\frac{\pi}{\lambda_1} - \frac{\pi}{\lambda_2}\right)x\right]\cos\left[\left(\frac{\pi}{\lambda_1} + \frac{\pi}{\lambda_2}\right)x\right]$$

Envelope:

$$k_{low} = (k_1 - k_2)/2$$

Carrier:

$$k_{high} = (k_1 + k_2)/2$$



Beating of two waves

Total:

$$y_{tot}(x) = 2A\cos\left[\left(\frac{\pi}{\lambda_1} - \frac{\pi}{\lambda_2}\right)x\right]\cos\left[\left(\frac{\pi}{\lambda_1} + \frac{\pi}{\lambda_2}\right)x\right]$$

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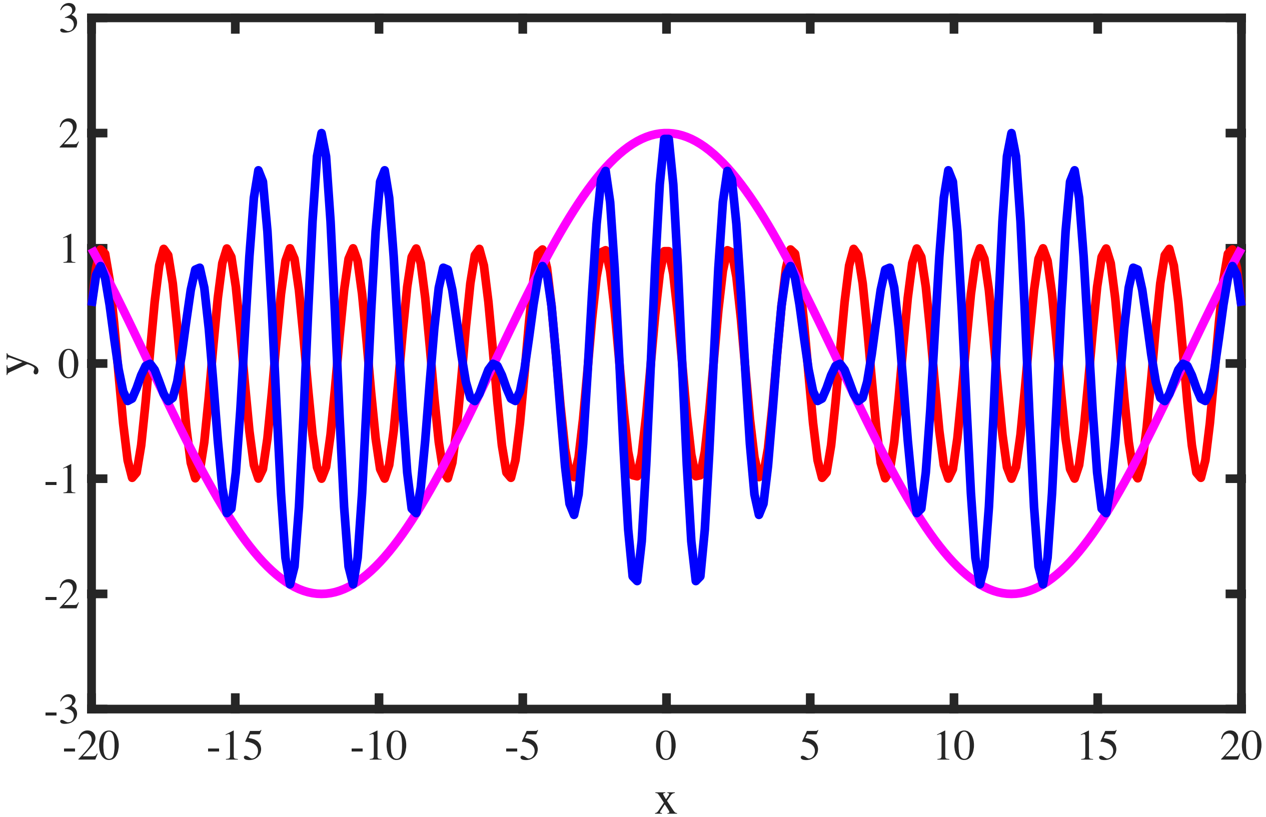


Beating of two waves

Total:

$$y_{tot}(x) = 2A\cos\left[\left(\frac{\pi}{\lambda_1} - \frac{\pi}{\lambda_2}\right)x\right]\cos\left[\left(\frac{\pi}{\lambda_1} + \frac{\pi}{\lambda_2}\right)x\right]$$

Envelope:

$$k_{low} = (k_1 - k_2)/2$$

Carrier:

$$k_{high} = (k_1 + k_2)/2$$

Note: The math above simply gives us a product of two cosines. Which we call carrier and which envelope, simply is decided by which one has the much larger wavelength (that cosine is called envelope)

2.3.2) Fourier decomposition

We managed to make wave more “pulsey” by adding two.
We can **perfectly** form **any** function if we take more waves:

Fourier theorem:

Any even function f(x) can be written as:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \;\; \tilde{g}(k) \;\; \cos(kx) \qquad (42)$$

If f(x) is periodic with period L

$$f(x) = \sum_{n=0}^{\infty} g_n \cos\left(\frac{2\pi n}{L} x\right) \qquad (43)$$

Fourier decomposition

Fourier theorem:

Any even function f(x) can be written as:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \;\; \tilde{g}(k) \;\; \cos(kx) \qquad (42)$$

If f(x) is periodic:

$$f(x) = \sum_{n=0}^{\infty} g_n \cos\left(\frac{2\pi n}{L} x\right) \qquad (43)$$

The coefficients g_n can be found via:

$$g_n = \frac{2}{L} \int_{-L/2}^{L/2} dx \;\; f(x) \cos\left(\frac{2\pi n}{L} x\right) \qquad (44)$$

BONUS MATERIAL: Fourier decomposition

Of course it also works for odd functions f(x)=-f(-x), using *sines*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \;\; \tilde{g}(k) \;\; \sin(kx) \qquad (45)$$

$$f(x) = \sum_{n=0}^{\infty} h_n \sin\left(\frac{2\pi n}{L} x\right) \qquad (46)$$

$$h_n = \frac{2}{L} \int_{-L/2}^{L/2} dx \;\; f(x) \sin\left(\frac{2\pi n}{L} x\right) \qquad (47)$$

BONUS: Fourier decomposition

More generally **any** function f(x) can be written as:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \ \tilde{f}(k) \, e^{ikx} \tag{48}$$

with

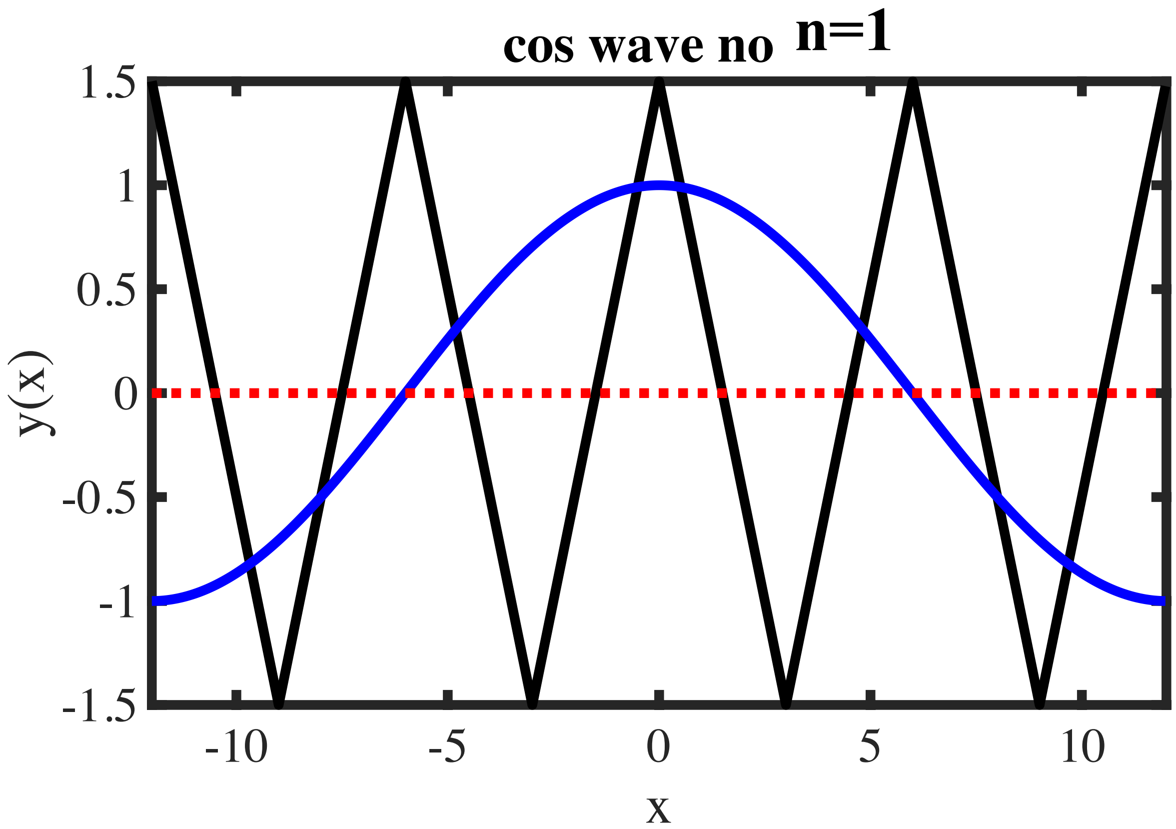
$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \ f(x) \, e^{-ikx} \tag{49}$$

using

$$e^{ikx} \equiv \cos(kx) + \underbrace{i}_{=\sqrt{-1}} \sin(kx) \tag{50}$$

Fourier decomposition

Example: Sawtooth curve (see fourier_decomposition_v1.m)



Legend

$f(x)$

$\cos\left(\frac{2\pi n}{L}x\right)$

$\sum_{m=0}^{n} g_n \cos\left(\frac{2\pi n}{L}x\right)$

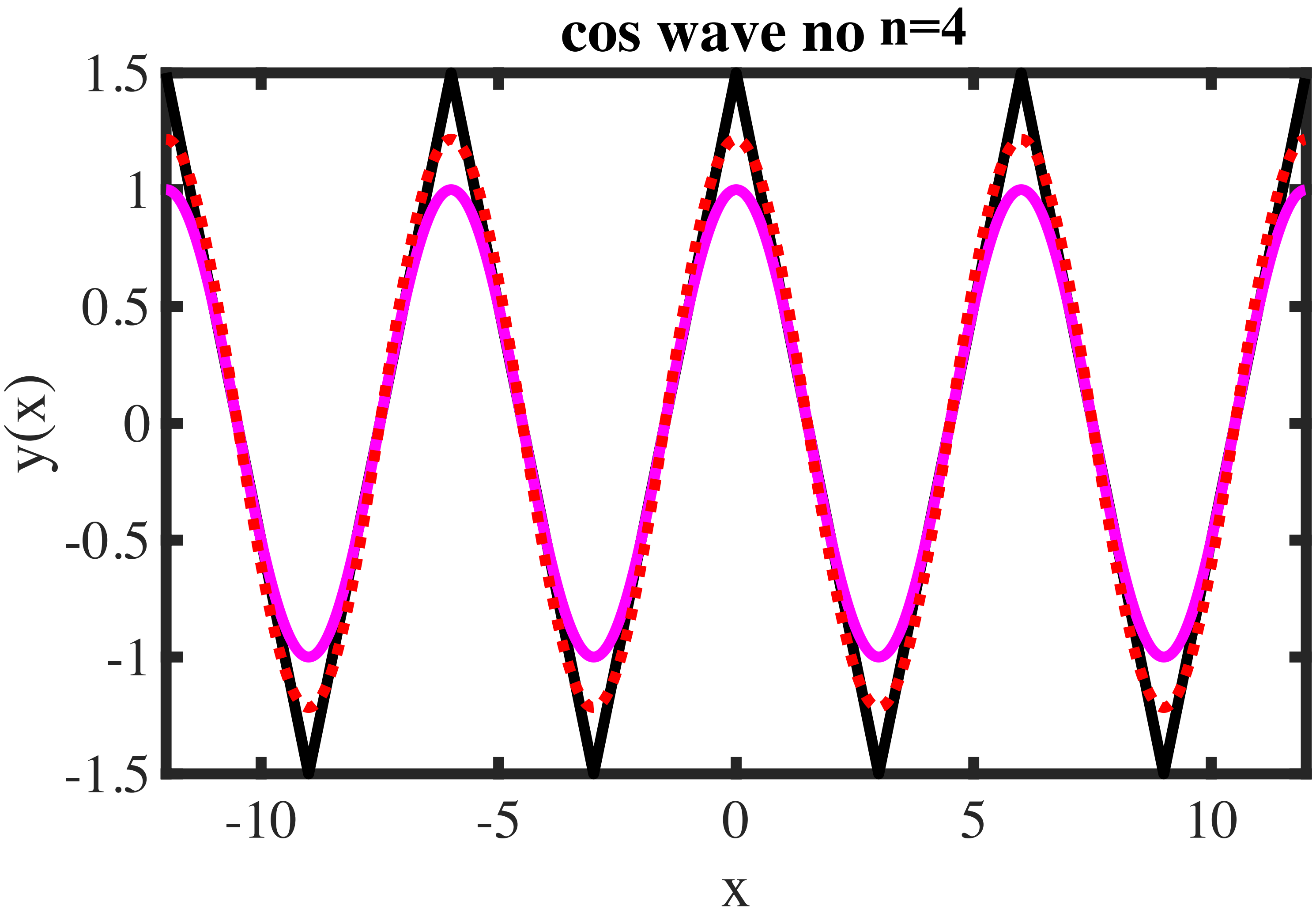
we use Eq. (43)

Discard this (g_n small), since wavelength wrong

See webapps in tutorial 5, for this!!!

Fourier decomposition

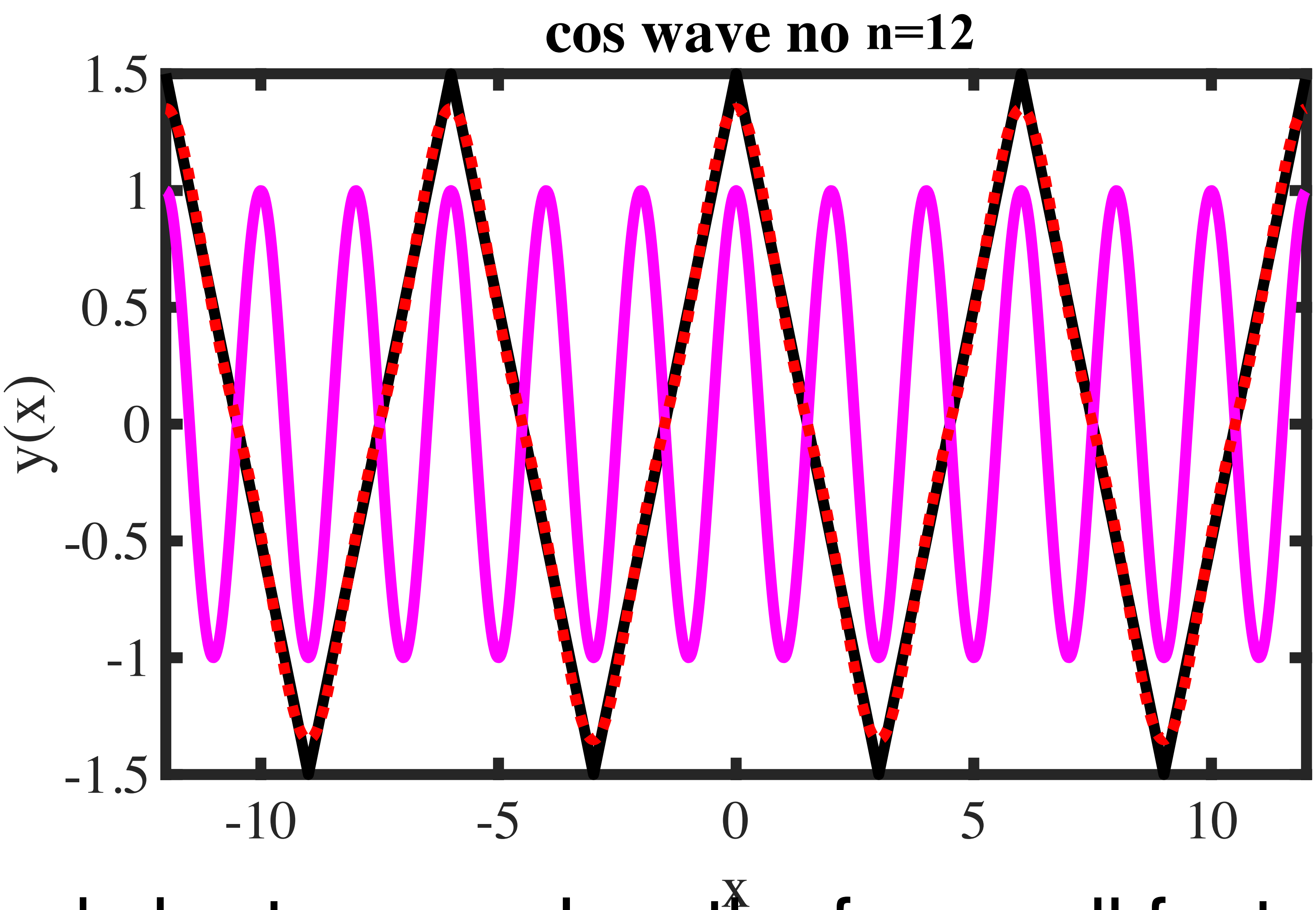
Example: Sawtooth curve



Keep this (g_n large), since wavelength good

Fourier decomposition

Example: Sawtooth curve



Need shorter wavelengths for small features

To see the complete animation,
follow this link: http://home.iiserb.ac.in/~sebastian/material/QuantPhys/fourier_sawtooth.mp4

Legend: (see Eq. 43 and 44)

black line $f(x) =$ sawtooth function

red dashed

cumulative sum $$s_n(x) = \sum_{m=0}^{n} g_m \cos\left(\frac{2\pi m}{L} x\right)$$

blue **or** **magenta**

$g_n < 10^{-5}$ $g_n > 10^{-5}$

trial cosine $$\cos\left(\frac{2\pi n}{L} x\right)$$

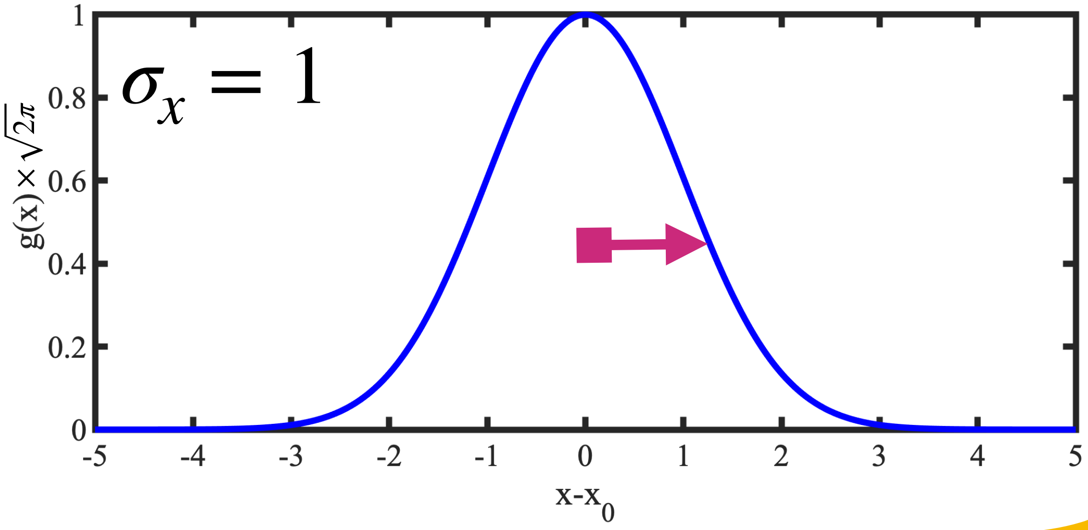
2.3.3) Gaussian wave packet

We call the combination of **many waves** a **wave packet**

A neat case is the **Gaussian wave packet**

Math: Gaussian function

$$g(x) = \frac{1}{\sqrt{2\pi}\boxed{\sigma_x}} e^{-\frac{(x-x_0)^2}{\boxed{2\sigma_x^2}}}$$



- σ_x is called the **width** (or **standard deviation**) of the Gaussian

- Pre-factor fixes **normalisation**

$$\int_{-\infty}^{\infty} dx \; g(x) = 1$$

Gaussian wave packet

We call the combination of **many waves** a **wave packet**

A neat case is the

Gaussian wave packet

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \;\; \tilde{g}(k) \;\; \cos(kx) \qquad \text{repeat (42)}$$

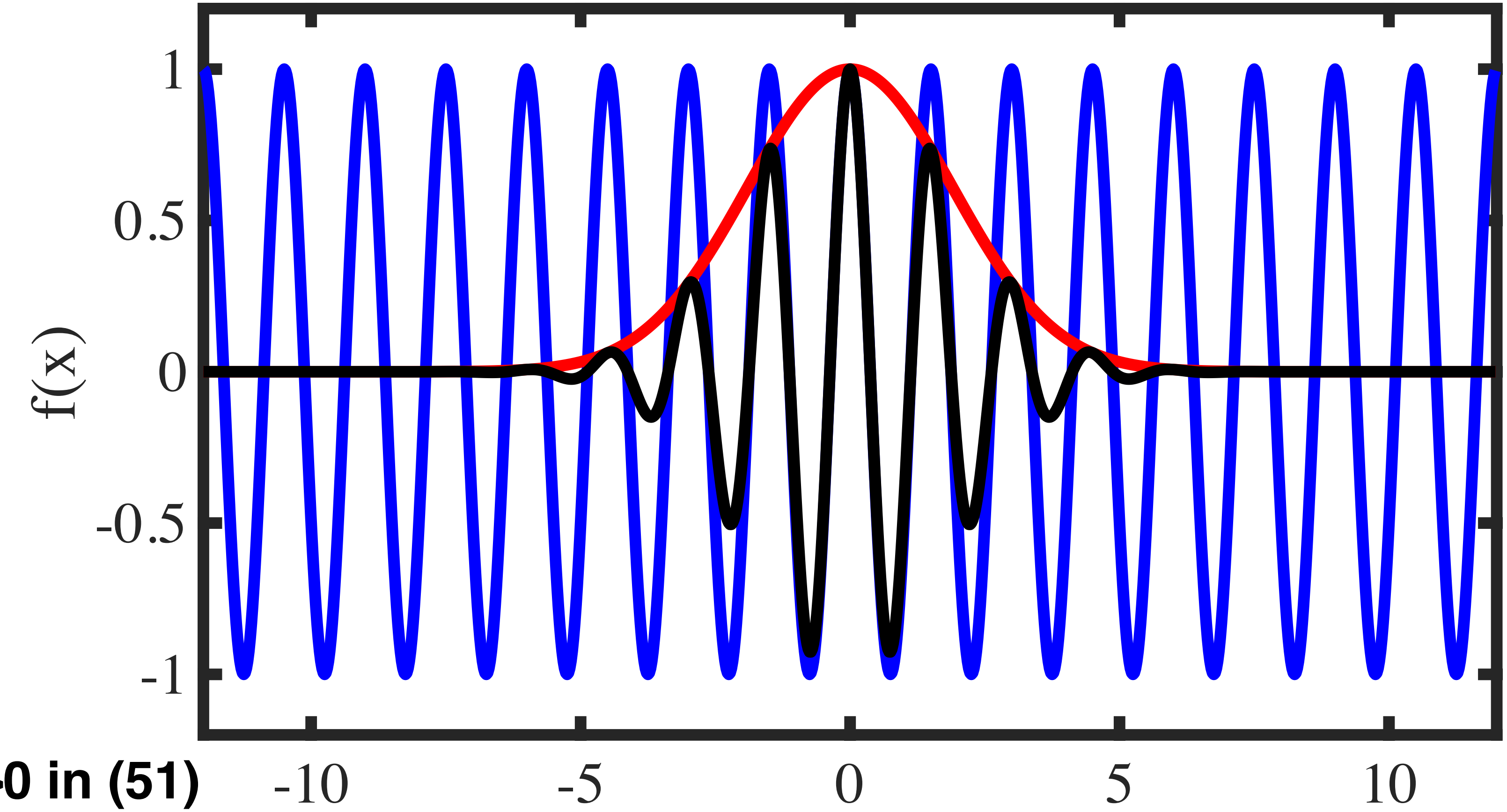
$$\tilde{g}(k) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \qquad (51)$$

Gaussian wave packet

$$f(x) \sim \int_{-\infty}^{\infty} dk \;\; e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \;\; \cos(kx)$$

$$f(x) \sim e^{-\frac{x^2}{2\sigma_x^2}} \cos(k_0 x)$$

Envelope of Gaussian-w.p. is again Gaussian:



This uses $k_0>0$ in (51)

To see the complete animation, follow this link: http://home.iiserb.ac.in/~sebastian/material/QuantPhys/fourier_gaussian_carrier.mp4

Legend: (see Eq. 43 and 44)

black line $$f(x) \sim e^{-\frac{x^2}{2\sigma_x^2}} \cos(k_0 x)$$

red dashed

cumulative sum $$s_n(x) = \sum_{m=0}^{n} g_m \cos\left(\frac{2\pi m}{L} x\right)$$

blue or **magenta** trial cosine $$\cos\left(\frac{2\pi n}{L} x\right)$$

$g_n < 10^{-5}$ $g_n > 10^{-5}$

Gaussian wave packet (math version)

factor to give overall scale of function

$$f(x) = \mathcal{N} \int_{-\infty}^{\infty} dk \; e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \cos(kx)$$

We can solve the above integral and find:

$$f(x) = \mathcal{N}\sqrt{2\pi}\sigma_k e^{-\frac{x^2\sigma_k^2}{2}} \cos(k_0 x)$$

We can write this as

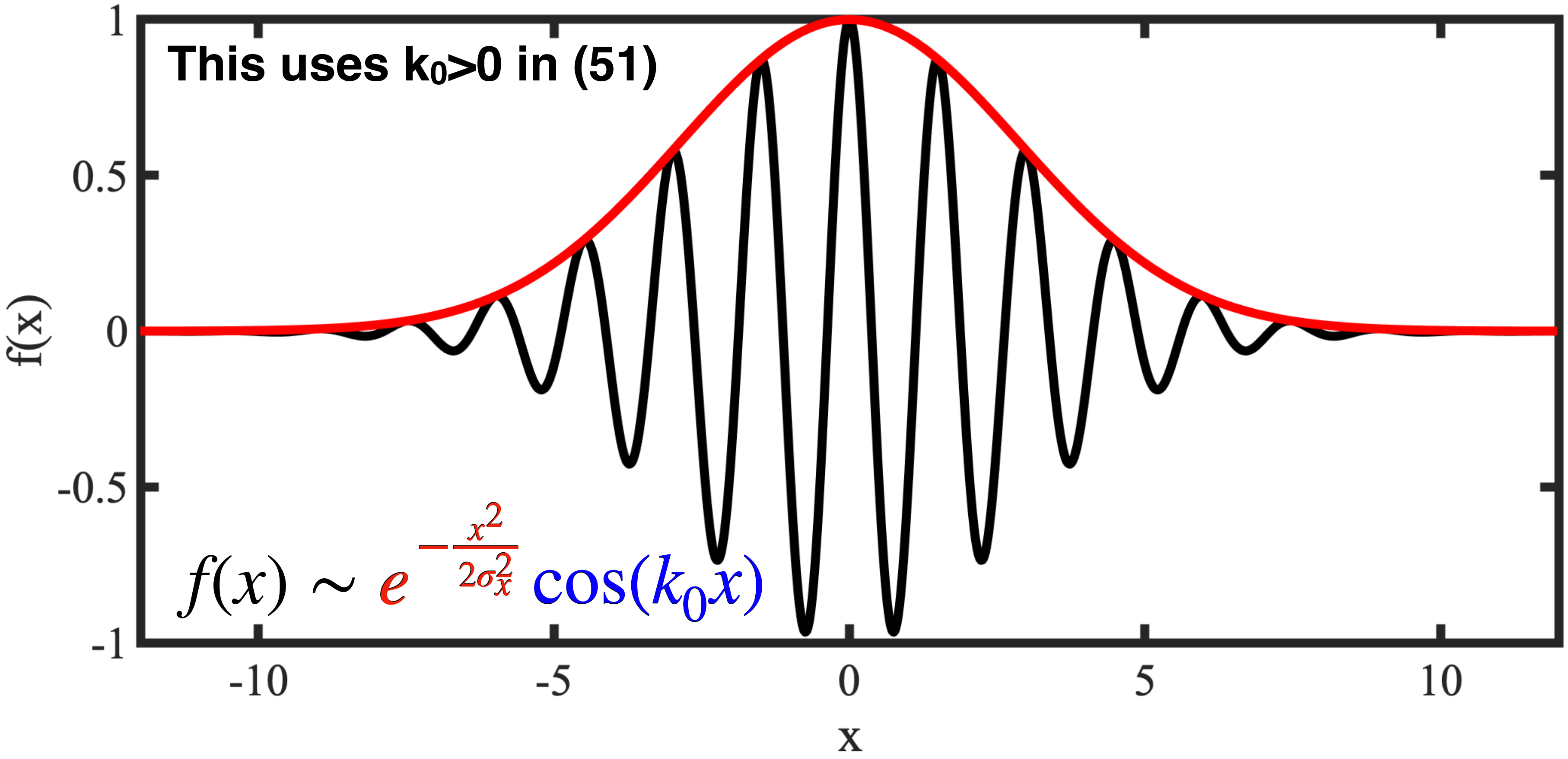
$$f(x) = \tilde{\mathcal{N}} e^{-\frac{x^2}{2\sigma_x^2}} \cos(k_0 x) \quad \text{with} \quad \sigma_x = 1/\sigma_k$$

…as we did on the previous slide.

Gaussian wave packet

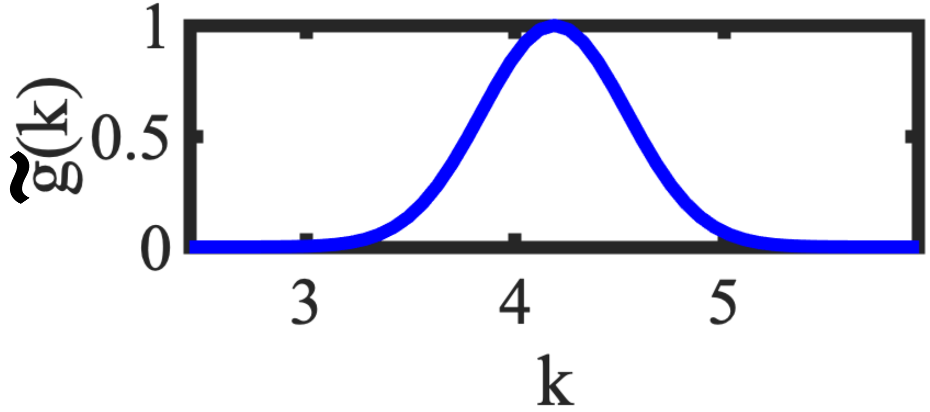
$$f(x) \sim \int_{-\infty}^{\infty} dk \; e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \; \cos(kx)$$

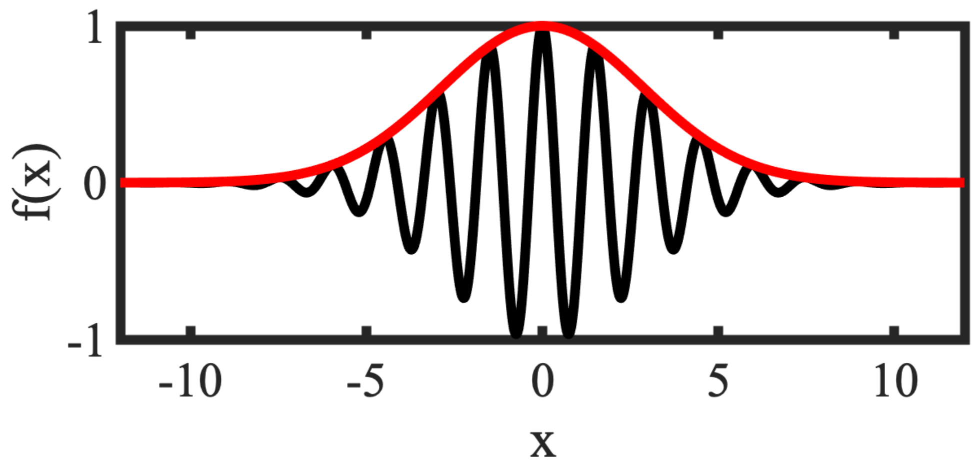
Envelope of Gaussian-w.p. is again Gaussian:



Gaussian wave packet

$$f(x) \sim \int_{-\infty}^{\infty} dk \;\, e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \;\cos(kx)$$



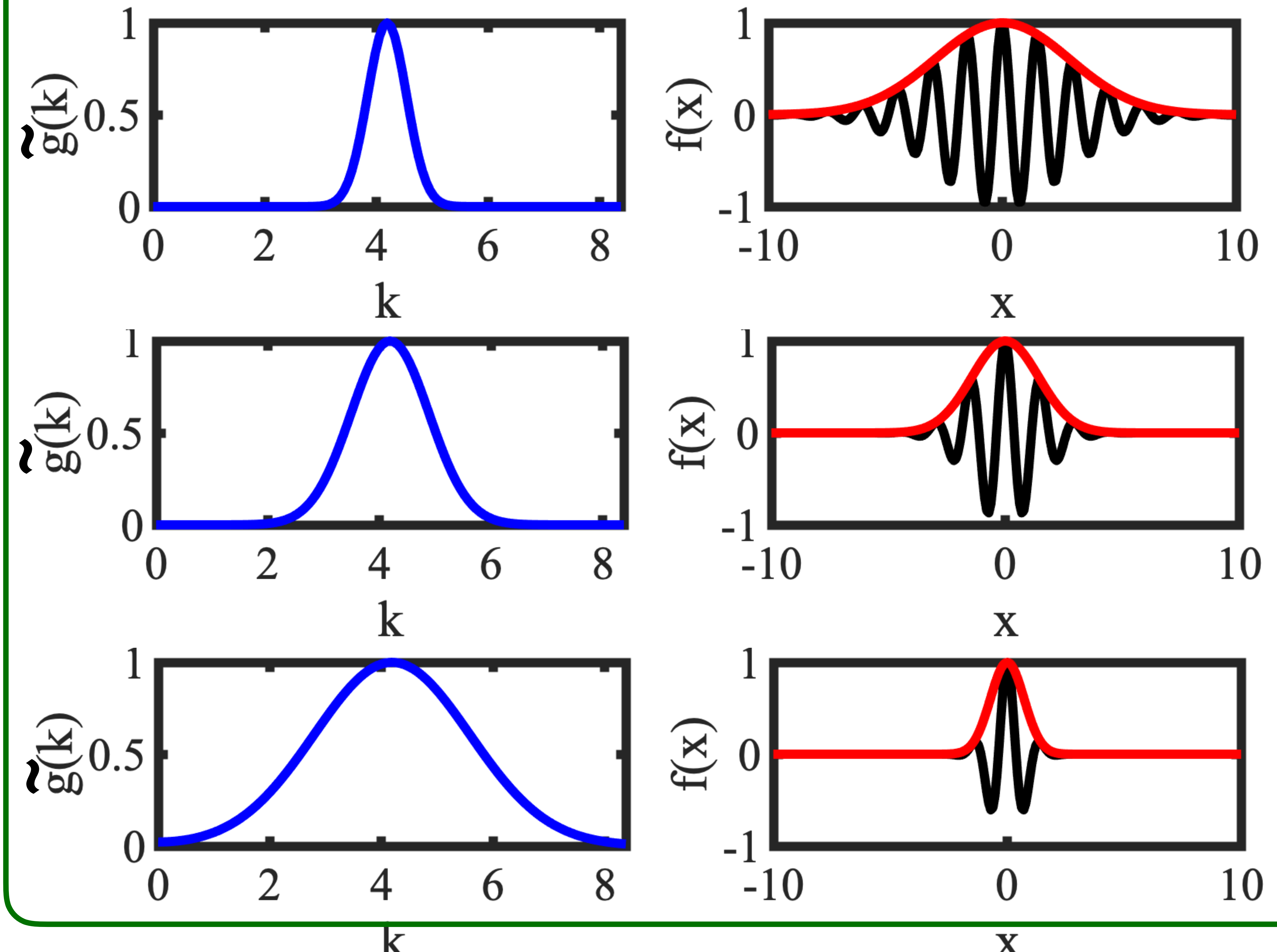


Gaussian: x and k **widths** are **inverse**

$$\sigma_x = 1/\sigma_k \tag{52}$$

- If we want a **more localised** wave packet, we need a **larger range of wave lengths!!!**

Examples:



2.3.4) Dispersion

Dispersion For waves in a medium, the phase velocity V may depend on the wave frequency ω.

- In other words, the relation between ω and k is not **proportional** (as in $\omega = Vk$)

- Then phase velocity $V = \omega/k$ is not constant

We call the dependence of ω on k
Dispersion relation $\omega = f(k)$
some function f

Dispersion

- Wave Eqn. (13) predicts equal phase velocity V=const. for all waves

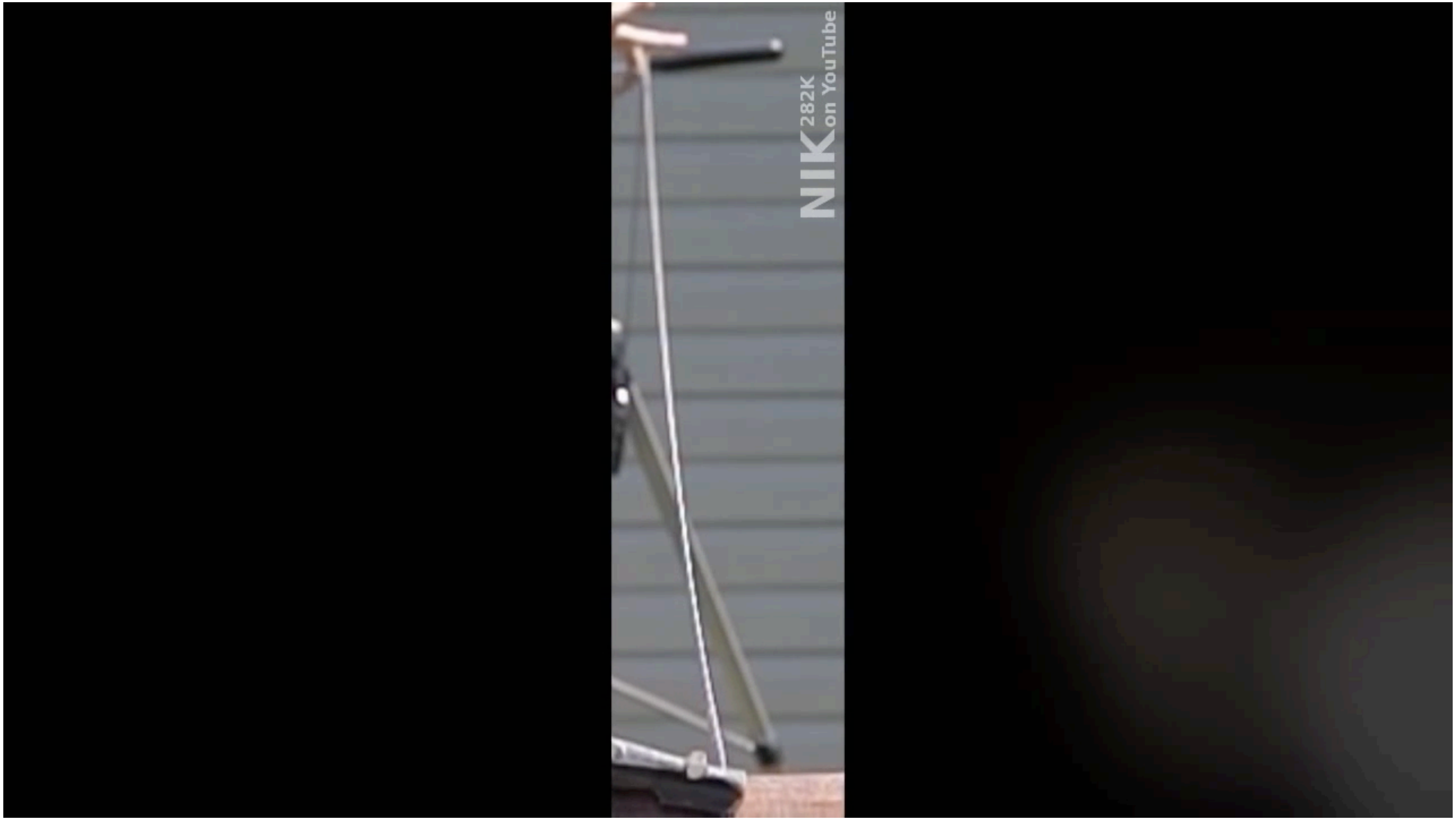
$$\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{V^2} \frac{\partial^2}{\partial t^2} y(x,t)$$

- Thus if we have dispersion it needs modification, e.g.

$$\frac{\partial^2}{\partial x^2} y(x,t) - \alpha \frac{\partial^4}{\partial x^4} y(x,t) = \frac{1}{\beta^2} \frac{\partial^2}{\partial t^2} y(x,t)$$

(I don’t tell you what α,β are, this is just an example for the mathematical structure)

Dispersion

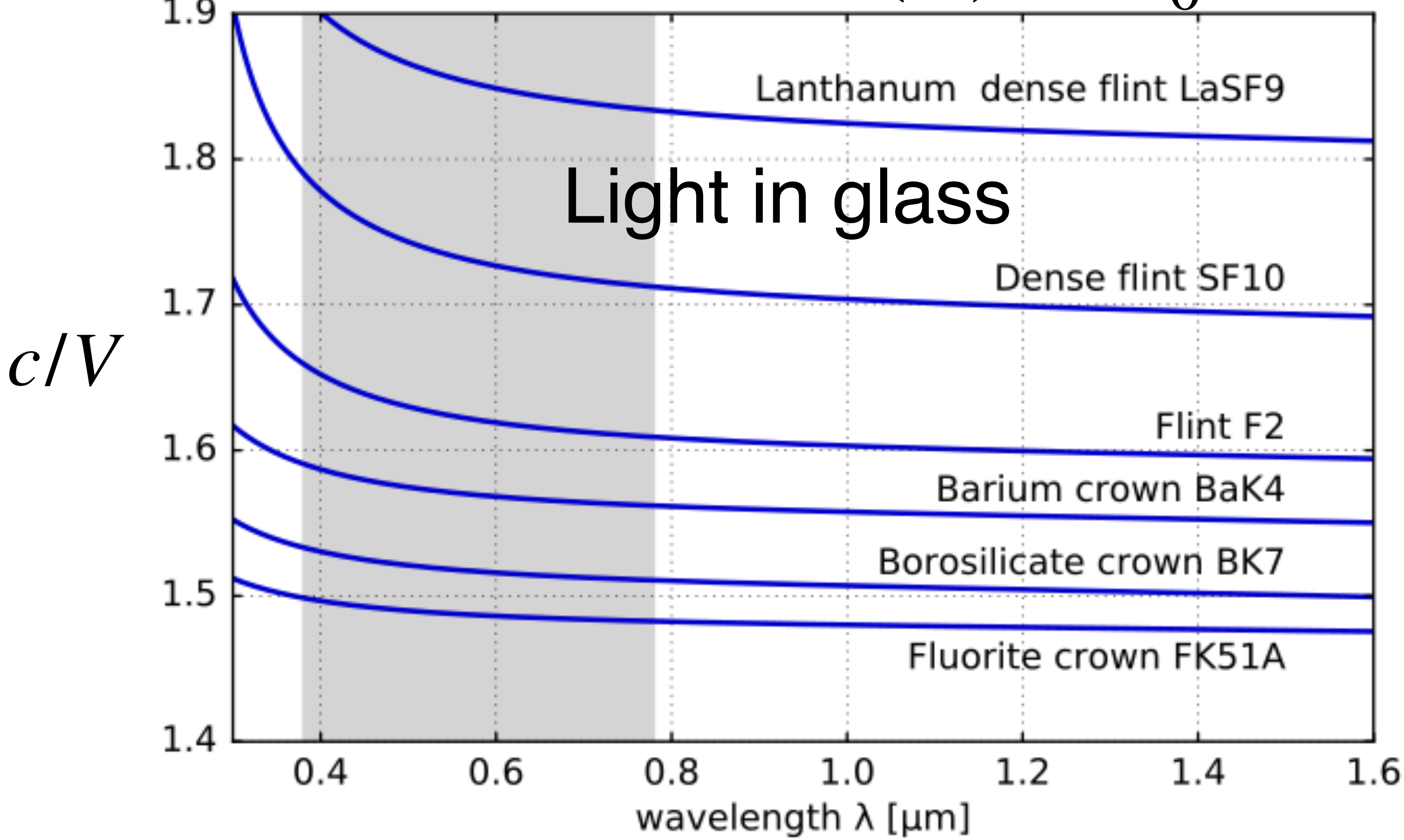


https://www.youtube.com/watch?v=KbmOcT5sX7I

Dispersion

Example:

- The dependence of phase velocity on frequency/ wavenumber is often weak i.e. $V(\omega) \approx V_0 \quad \forall\omega$



2.3.5) Group velocity

Consider Gaussian wave-packet

$$f(x) \sim \int_{-\infty}^{\infty} dk \; e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \cos(kx)$$

Now lets make waves moving

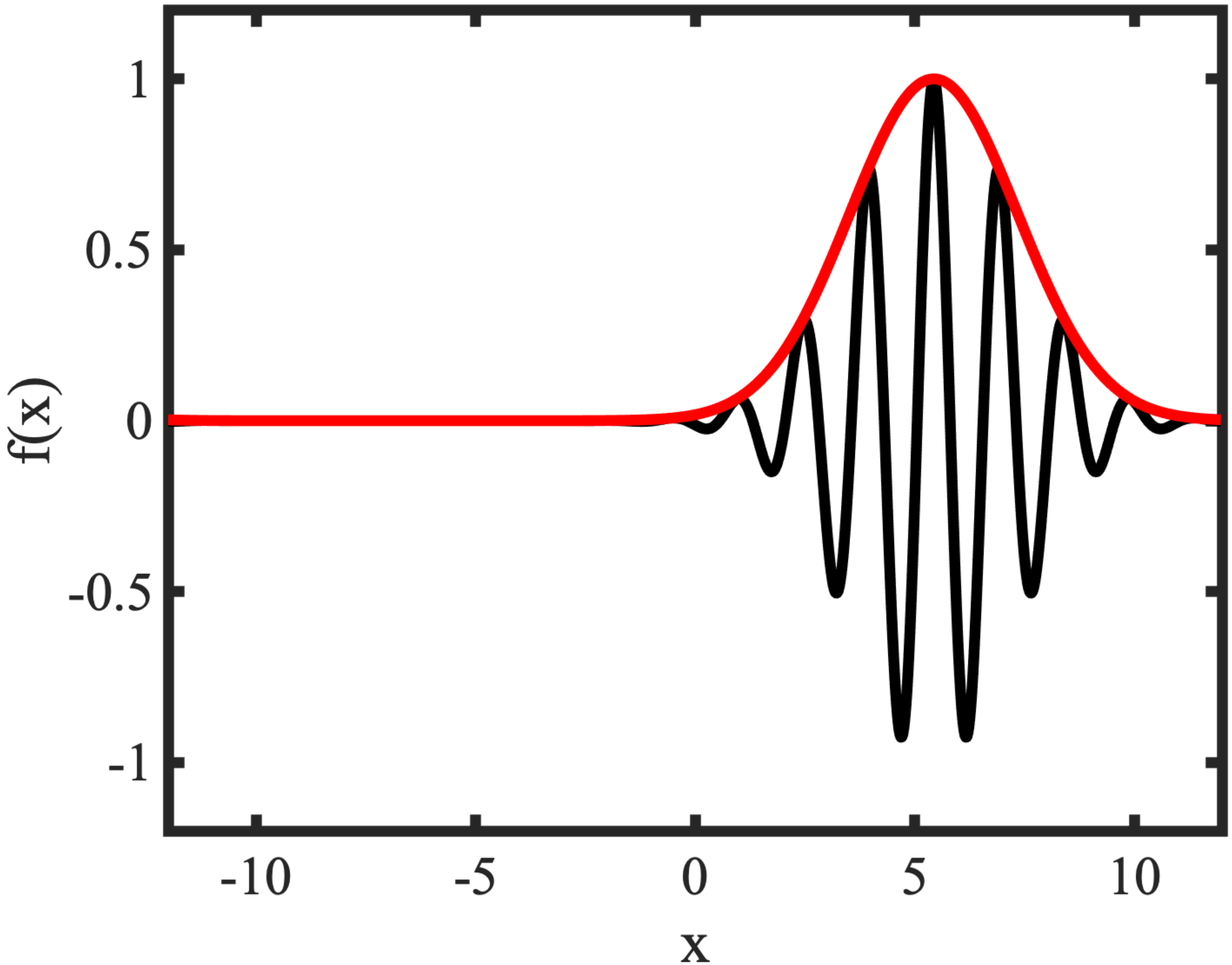
$$f(x,t) \sim \int_{-\infty}^{\infty} dk \; e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \underline{\cos(kx - \omega t)}$$

$$\cos[k(x - Vt)]$$

If no dispersion: $\omega = Vk$ (see Eq. 8, same V)

Group velocity

Moving Gaussian wave-packet without dispersion



2.3.5) Group velocity

Consider Gaussian wave-packet

$$f(x) \sim \int_{-\infty}^{\infty} dk \;\; e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \;\; \cos(kx)$$

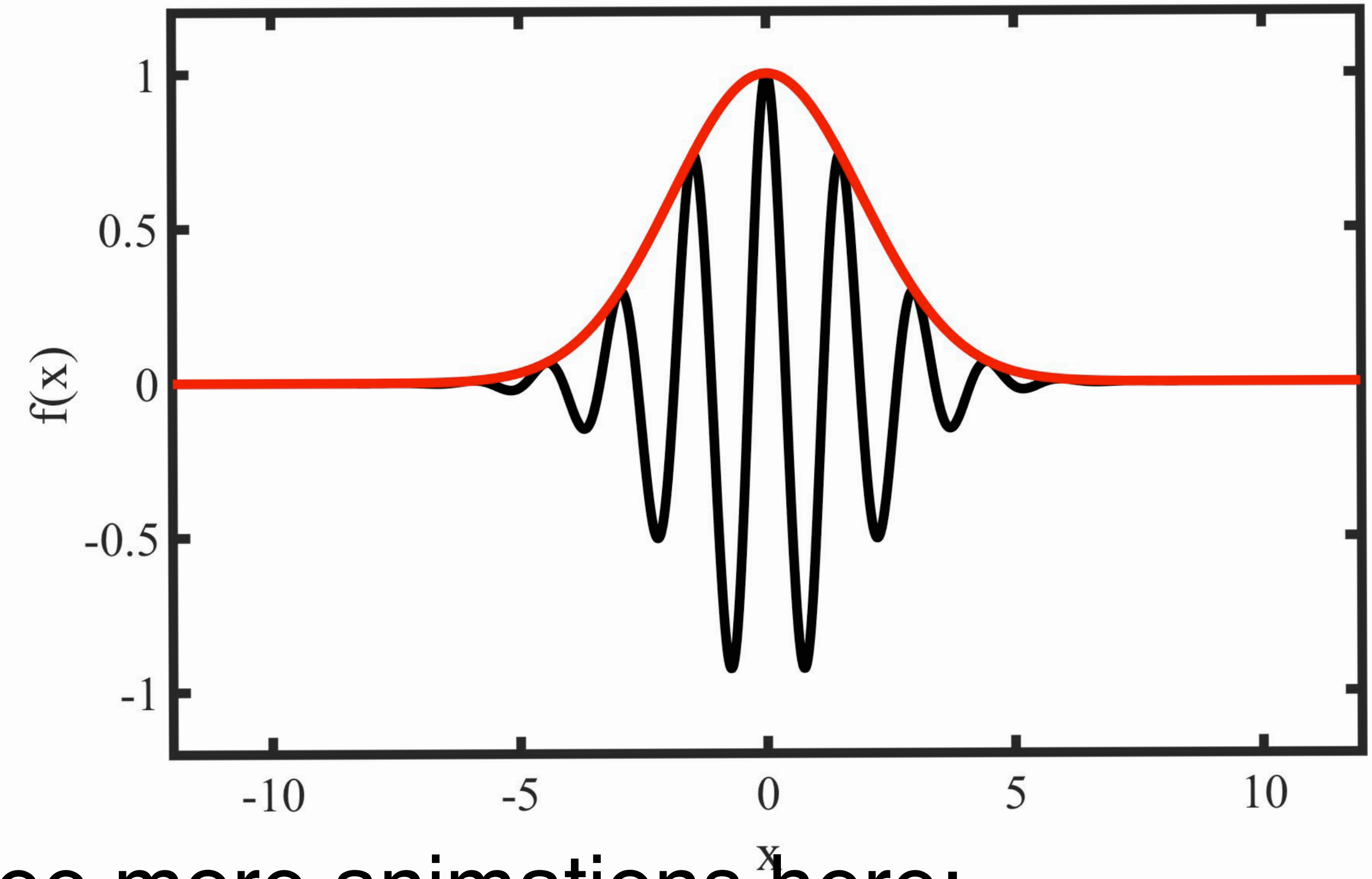
Now lets make waves moving

$$f(x,t) \sim \int_{-\infty}^{\infty} dk \;\; e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \;\; \cos[k(x-\underline{V(k)}t)]$$

If dispersion: Different velocities $V(k)$
for different "k" parts of wavepacket

Group velocity

Moving Gaussian wave-packet

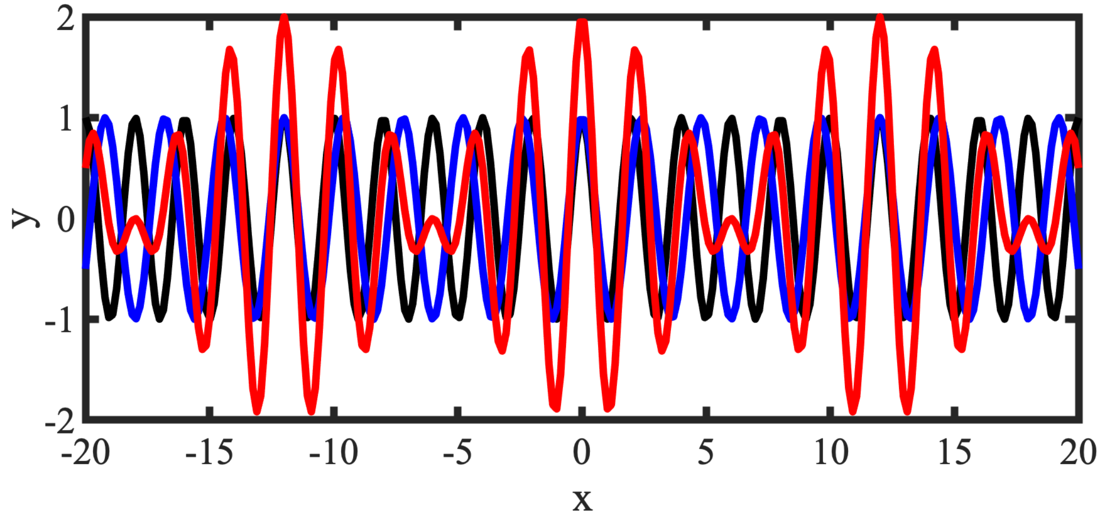


see more animations here:

https://blog.soton.ac.uk/soundwaves/further-concepts/2-dispersive-waves/

Simpler example: Motion of beating waves

Go back to beating example



Assume two different waves

$$y_1(x,t) = A\cos[(\omega - \Delta\omega/2)t - (k - \Delta k/2)x]$$

$$y_2(x,t) = A\cos[(\omega \underline{+} \Delta\omega/2)t - (k \underline{+} \Delta k/2)x]$$

second has only slightly different ω and k.

Motion of beating waves

Add them as in section 2.3.1.)

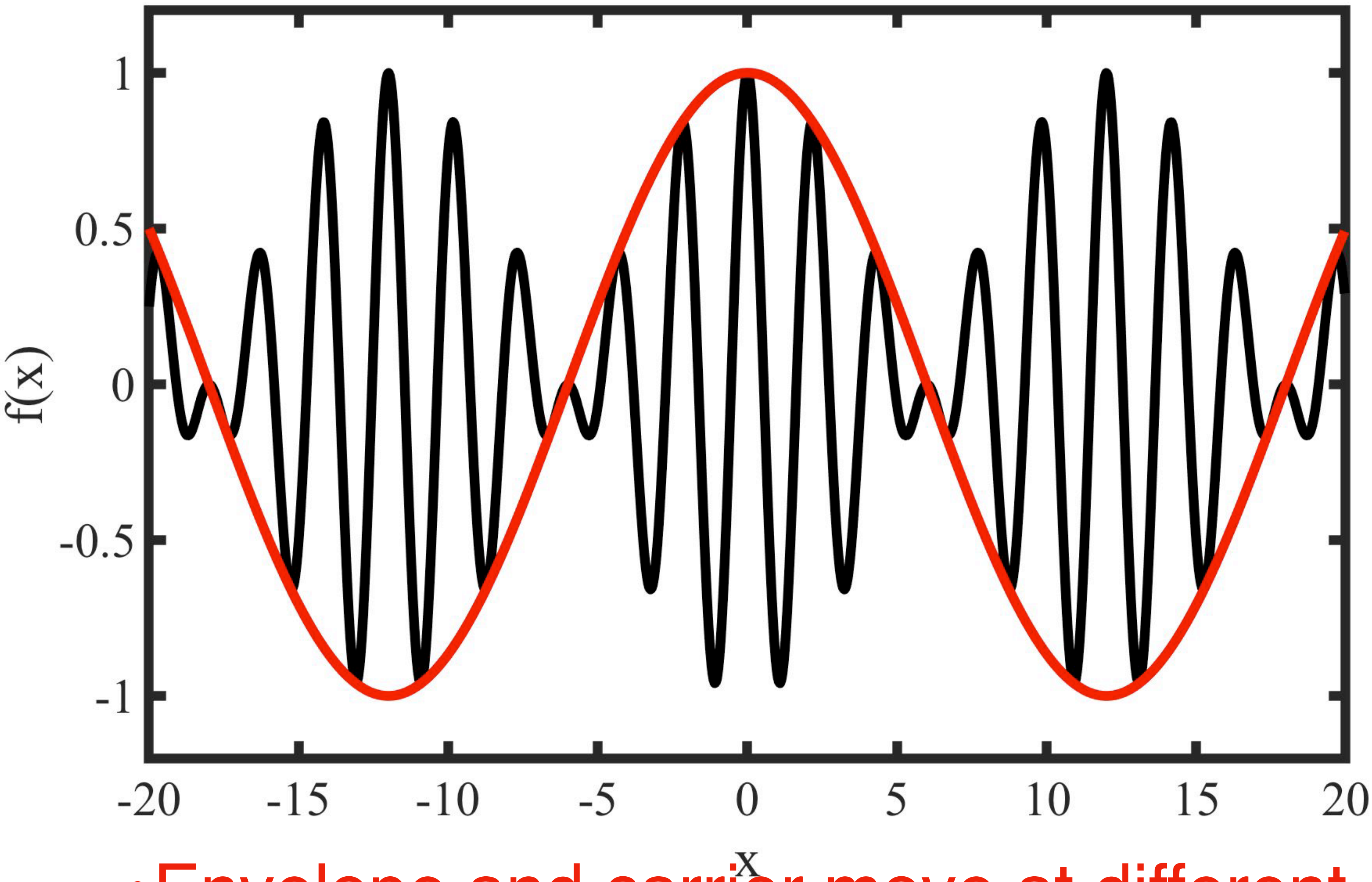
$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$= \underbrace{2A\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)}_{\text{Moving Envelope:}}\ \underbrace{\cos(\omega t - kx)}_{\text{Moving Carrier:}} \tag{53}$$

Moving Envelope: **Moving** Carrier:

- See book for details.

Movie: **Motion of beating waves**



- Envelope and carrier move at different velocity!!

Motion of beating waves

$$y = 2A\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)\cos(\omega t - kx)$$

Moving Carrier: **phase** velocity

Moving Envelope:

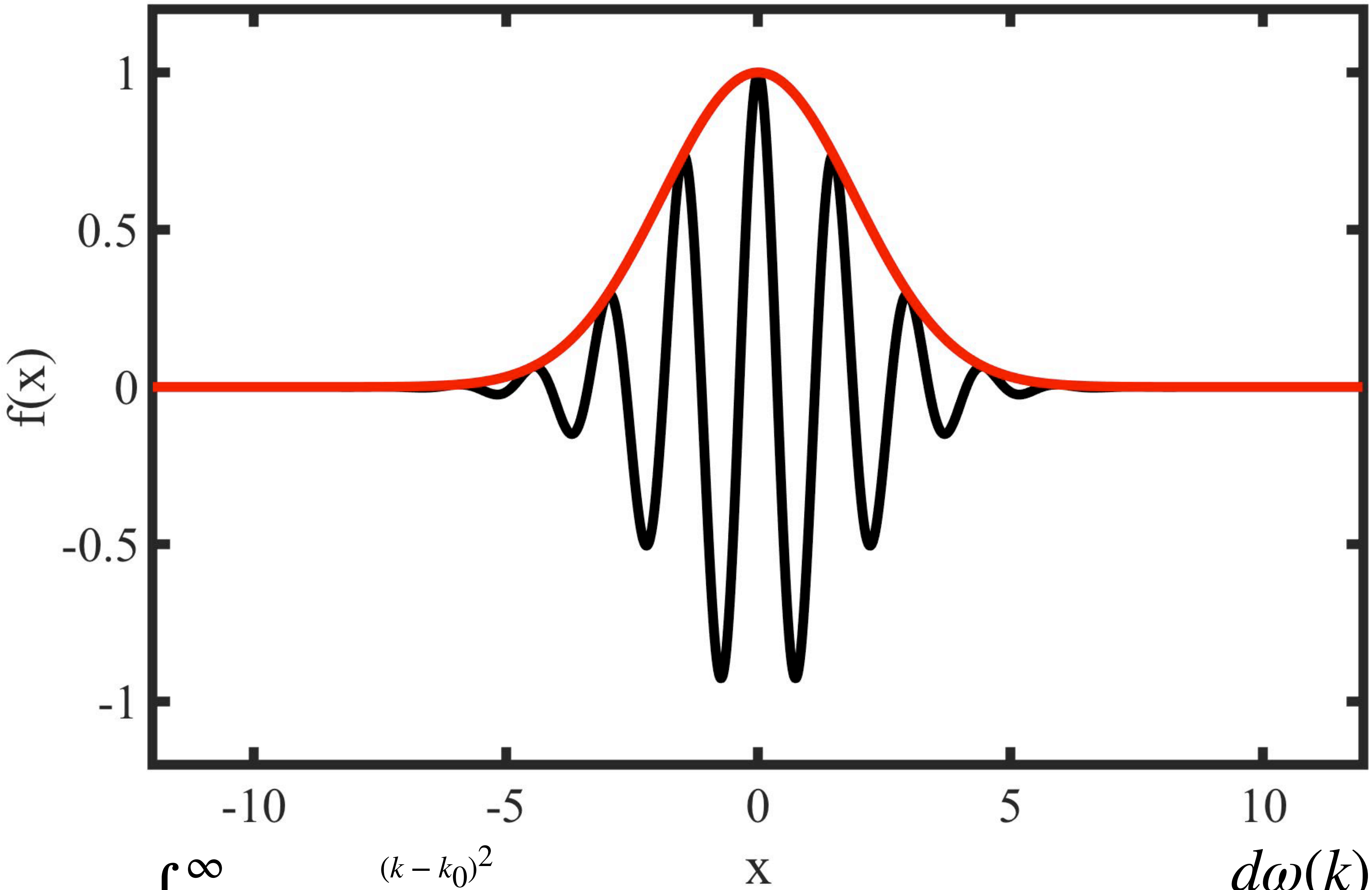
Using Eq. (8) we infer for motion of envelope

Group velocity

$$v_g = \frac{\Delta\omega}{\Delta k}$$ (Two waves) **(53)**

$$v_g = \frac{d\omega}{dk}$$ (Many waves) **(54)**

Movie: **Gaussian moving wavepacket**



$$f(x) \sim \int_{-\infty}^{\infty} dk \ e^{-\frac{(k-k_0)^2}{2\sigma_k^2}} \cos(kx)$$

$$v_g = \frac{d\omega(k)}{dk}\bigg|_{k=k_0}$$