PHY 304, II-Semester 2023/24, Tutorial 8

19. April. 2024

Work in the same teams as for assignments. Do "Stages" in the order below. Discuss on your table in AIR. When all on your table finished a stage, make sure all students at your table understand the solution and agree on one by using the board.

Stage 1 **Time evolution pictures:** For week 10 we need to revise QM-1, section 3.9, on different pictures of time-evolution. Remind yourself of those. We will first consider the example of a quantum harmonic oscillator, for which we can write the Hamiltonian as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{1}$$

in terms of ladderoperators for which $\hat{a}^\dagger|\,\varphi_n\,\rangle = \sqrt{n+1}|\,\varphi_{n+1}\,\rangle$, $\hat{a}|\,\varphi_n\,\rangle = \sqrt{n}|\,\varphi_{n-1}\,\rangle$ and $\left[\hat{a},\hat{a}^\dagger\right] = 1$ holds. (we had called these $\hat{a}_\pm$ in example 20 of Qm-I).

(i) Consider the initial state $|\,\Psi(t=0)\,\rangle = (|\,\varphi_0\,\rangle + |\,\varphi_1\,\rangle)/\sqrt{2}$. Write the time evolving state $|\,\Psi(t)\,\rangle$ in the Schrödinger picture and from that calculate the expectation value $\langle\,\hat{x}\,\rangle(t)$ using $\hat{x} = (\hat{a}^\dagger + \hat{a})\sqrt{\hbar/(2m\omega)}$.

(ii) Now, find the time-evolution equation for the operator $\hat{a}(t)$ in the Heisenberg picture. Solve it, and then calculate $\langle\,\hat{x}\,\rangle(t)$ again in the Heisenberg picture (remember here quantum states are time-independent, so $|\,\Psi\,\rangle = (|\,\varphi_0\,\rangle + |\,\varphi_1\,\rangle)/\sqrt{2}$ is the "eternal" state.

(iii) Now let's change the quantum system to a spin-1/2 with Hamiltonian

$$\hat{H} = \frac{\hbar\Omega}{2}\hat{\sigma}_x. \tag{2}$$

Show that the time evolution operator is

$$\hat{U}(t) = \begin{bmatrix} \cos(\Omega t/2) & -i\sin(\Omega t/2) \\ -i\sin(\Omega t/2) & \cos(\Omega t/2) \end{bmatrix} \tag{3}$$

Hint: Note that $\hat{\sigma}_x^2 = \mathbb{1}$, $\hat{\sigma}_x^3 = \hat{\sigma}_x$, $\hat{\sigma}_x^4 = \mathbb{1}$ etc. Use the power series for exp, sin, cos *and then plug it all together.*

(iv) From that, find the time evolving state from initial state $|\,\Psi(t=0)\,\rangle = |\downarrow\rangle$ in the Schrödinger picture and compare the probability $p_\uparrow$ to be in $|\uparrow\rangle$ with section 9.3.5.

(v) **Bonus:** To see this in the Heisenberg picture we define the projector $\hat{P}_\uparrow = |\uparrow\rangle\langle\uparrow|$ so that $p_\uparrow = \langle\,\hat{P}_\uparrow\,\rangle$. Then find $\langle\,\hat{P}_\uparrow(t)\,\rangle$, with $\hat{P}(t) = \hat{U}^\dagger(t)\hat{P}_\uparrow\hat{U}(t)$.

Stage 2 Interaction picture Consider the time dependent two level system (spin-1/2) that we had looked at in example 71 with Hamiltonian in matrix form

$$\underline{\underline{H}} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}}_{=\hat{H}^{(0)}} + \underbrace{\begin{pmatrix} 0 & \kappa(t) \\ \kappa(t) & 0 \end{pmatrix}}_{=\hat{H}'(t)}. \tag{4}$$

with the same $\kappa(t)$ as used there.

(i) Find the time evolution of the operator $\hat{S}_x = \hbar(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)$ in the interaction picture.

(ii) Write the interaction picture state in the usual spin basis

$$|\Psi_I(t)\rangle = c_{I,\uparrow}(t)|\uparrow\rangle + c_{I,\downarrow}(t)|\downarrow\rangle, \tag{5}$$

and find the evolution equation for the time dependent coefficients

(iii) BONUS: Using both of the above, find the time evolution of the expectation value $\langle \hat{S}_x \rangle(t)$ in the interaction picture, starting from the initial state $|\Psi(t=0)\rangle = |\uparrow\rangle$ (might need mathematica).

Stage 3 Higher-order Time dependent perturbation theory

(i) How do we move to higher-order time dependent perturbation theory and what is the physical picture of the structure of quantum dynamics that it suggests to us?