PHY 304, II-Semester 2023/24, Tutorial 8

19. April. 2024

Work in the same teams as for assignments. Do "Stages" in the order below. Discuss on your table in AIR. When all on your table finished a stage, make sure all students at your table understand the solution and agree on one by using the board.

Stage 1 Time evolution pictures: For week 10 we need to revise QM-1, section 3.9, on different pictures of time-evolution. Remind yourself of those. We will first consider the example of a quantum harmonic oscillator, for which we can write the Hamiltonian as

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{1}$$

in terms of ladderoperators for which $\hat{a}^\dagger|\,\varphi_n\,\rangle = \sqrt{n+1}|\,\varphi_{n+1}\,\rangle$, $\hat{a}|\,\varphi_n\,\rangle = \sqrt{n}|\,\varphi_{n-1}\,\rangle$ and $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ holds. (we had called these $\hat{\hat{a}}_\pm$ in example 20 of Qm-I).

(i) Consider the initial state $|\,\Psi(t=0)\,\rangle = (|\,\varphi_0\,\rangle + |\,\varphi_1\,\rangle)/\sqrt{2}$. Write the time evolving state $|\,\Psi(t)\,\rangle$ in the Schrödinger picture and from that calculate the expectation value $\langle\,\hat{x}\,\rangle(t)$ using $\hat{x} = (\hat{a}^\dagger + \hat{a})\sqrt{\hbar/(2m\omega)}$.

Solution: We know the Schrödinger picture state since QM-I week2, Eqn. (1.70), using that here we can write $|\,\Psi(t=0)\,\rangle = c_0(0)|\,\varphi_0\,\rangle + c_1(0)|\,\varphi_1\,\rangle)$, with $c_0(0) = c_1(0) = 1/\sqrt{2}$ hence

$$|\,\Psi(t)\,\rangle = (e^{-iE_0t/\hbar}|\,\varphi_0\,\rangle + e^{-iE_1t/\hbar}|\,\varphi_1\,\rangle)/\sqrt{2} \tag{2}$$

we also know $E_k = \hbar\omega(k + 1/2)$, which can be inserted above. Note that "Schrödinger picture" only means "the state is evolving, not the operators". It does not mean you have to use a time-evolution operator as in $|\,\Psi(t)\,\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\,\Psi(0)\,\rangle$. However you can easily see that this is fully equivalent to Eqn. (1.70) which you know since long time. Let $|\,\Psi(0)\,\rangle = \sum_n c_n(0)|\,\phi_n\,\rangle$, where $\hat{H}|\,\phi_n\,\rangle = E_n|\,\phi_n\,\rangle$, then

$$|\,\Psi(t)\,\rangle = |\,\Psi(0)\,\rangle = \sum_n c_n(0)e^{-\frac{i}{\hbar}\hat{H}t}|\,\phi_n\,\rangle = \sum_n c_n(0)\left[\sum_{k=0}^{\infty}\frac{\left(-\frac{i}{\hbar}\hat{H}t\right)^k}{k!}\right]|\,\phi_n\,\rangle$$
$$= \sum_n c_n(0)\left[\sum_{k=0}^{\infty}\frac{\left(-\frac{i}{\hbar}E_nt\right)^k}{k!}\right]|\,\phi_n\,\rangle = \sum_n c_n(0)e^{-\frac{i}{\hbar}E_nt}|\,\phi_n\,\rangle, \tag{3}$$

which is Eqn. (1.70). We have first inserted the power series for exp *and then used $\hat{H}^k|\,\phi_n\,\rangle = (E_n)^k|\,\phi_n\,\rangle$ and then converted the power series back to an* exp*. In the future you do not need to do these steps anymore, you can directly tell that $e^{\hat{O}}|\,\phi\,\rangle = e^o|\,\phi\,\rangle$, if $|\,\phi\rangle$ is an eigenstate of $\hat{O}$ with eigenvalue o.*

Now to find

$$
\begin{aligned}
\langle \hat{x} \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \langle \hat{a}^\dagger + \hat{a} \rangle \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \left(e^{iE_0 t/\hbar}\langle \varphi_0 | + e^{iE_1 t/\hbar}\langle \varphi_1 |\right) \left(\hat{a}^\dagger + \hat{a}\right) \left(e^{-iE_0 t/\hbar}| \varphi_0 \rangle + e^{-iE_1 t/\hbar}| \varphi_1 \rangle\right) \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \left(e^{i(E_1 - E_0)t/\hbar}\langle \varphi_1 |\hat{a}^\dagger| \varphi_0 \rangle + e^{i(E_0 - E_1)t/\hbar}\langle \varphi_0 |\hat{a}| \varphi_1 \rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \cos[(E_1 - E_0)t/\hbar] = \sqrt{\frac{\hbar}{2m\omega}} \cos[\omega t]. \qquad (4)
\end{aligned}
$$

In the third equality we have already discarded two terms which, after applying ladder operators, will be zero due to orthogonality of the state. In the last equality we used $E_n = \hbar\omega(n + 1/2)$.

(ii) Now, find the time-evolution equation for the operator $\hat{a}(t)$ in the Heisenberg picture. Solve it, and then calculate $\langle \hat{x} \rangle(t)$ again in the Heisenberg picture (remember here quantum states are time-independent, so $| \Psi \rangle = (| \varphi_0 \rangle + | \varphi_1 \rangle)/\sqrt{2}$ is the "eternal" state.
Solution: To find the time-evolution of an operator in the Heisenberg picture, we either solve Heisenberg's equation (Eq. 3.69) or apply Eq. (3.66) [multiply from left and right with the time-evolution operator]. Here we can do the former:

$$
i\hbar\frac{d}{dt}\hat{a}_H(t) = \left[\hat{a}_H, \hat{H}_H\right] \overset{(i)}{=} \left[\hat{a}, \hat{H}\right] = \hbar\omega\left[\hat{a}, \hat{a}^\dagger\hat{a} + 1/2\right] = \hbar\omega \underbrace{\left[\hat{a}, \hat{a}^\dagger\right]}_{=1} \hat{a} \overset{(ii)}{=} \hbar\omega\hat{a}. \qquad (5)
$$

Comments about some items above: (i) For commutators that are a number (rather than an operator), we can show that

$$
\begin{aligned}
\left[\hat{a}_H, \hat{H}_H\right] &= \left[\hat{U}^\dagger(t)\hat{a}_S\hat{U}(t), \hat{U}^\dagger(t)\hat{H}_S\hat{U}(t)\right] \\
&= \hat{U}^\dagger(t)\hat{a}_S \underbrace{\hat{U}(t)\hat{U}^\dagger(t)}_{=\mathbb{1}} \hat{H}_S\hat{U}(t) - \hat{U}^\dagger(t)\hat{H}_S\hat{U}(t)\hat{U}^\dagger(t)\hat{a}_S\hat{U}(t) \\
&= \hat{U}^\dagger(t)\left[\hat{a}_S, \hat{H}_S\right]\hat{U}(t) \overset{comm.\_number}{=} \underbrace{\hat{U}^\dagger(t)\hat{U}(t)}_{=\mathbb{1}} \left[\hat{a}_S, \hat{H}_S\right] \qquad (6)
\end{aligned}
$$

(ii) We use $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, *which you can show from Eq. (2.45), taking into account the slightly different definition of ladder operators (see example 20). The* $\hat{a}$, $\hat{a}^\dagger$ *here are called* $\hat{\tilde{a}}_\pm$ *in that example. From now on, forget the earlier ones, always work with operators from this tutorial and their commutation relation.*
*We can easily solve (**??**) to give* $\hat{a}_H(t) = \hat{a}(0)e^{-i\omega t}$. *In this we have first ignored the fact that* $\hat{a}$ *is an operator and solved the ODE as we usually*

would, but then we can convince ourselves that it also holds for operators, since we can take matrix elements $\langle n | \cdots | m \rangle$ *with* any *states* $| n \rangle$, $| m \rangle$ *on both sides and find the equation to be correct.*
Calculating again the expectation value of position, but this time in the Heisenberg picture:

$$\begin{aligned}
\langle \hat{x} \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \langle \hat{a}(t)^\dagger + \hat{a}(t) \rangle \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \left(\langle \varphi_0 | + \langle \varphi_1 | \right) \left(\hat{a}^\dagger(t) + \hat{a}(t) \right) \left(| \varphi_0 \rangle + | \varphi_1 \rangle \right) \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \left(\langle \varphi_0 | + \langle \varphi_1 | \right) \left(\hat{a}^\dagger(0) e^{-i\omega t} + \hat{a}(0) e^{i\omega t} \right) \left(| \varphi_0 \rangle + | \varphi_1 \rangle \right) \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \left(\langle \varphi_0 | \hat{a}(0) | \varphi_1 \rangle e^{i\omega t} + \langle \varphi_1 | \hat{a}^\dagger(0) | \varphi_0 \rangle e^{-i\omega t} \right) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \cos[\omega t]
\end{aligned} \tag{7}$$

as we had seen before: **Warning:** *Since in the Heisenberg picture states do not evolve, and only at* $t = 0$ *co-incide with the Schrödinger picture state, you can only know how to apply ladder operators at* $t = 0$ *onto states. I.e.* $\hat{a}(0)| \varphi_1 \rangle = | \varphi_0 \rangle$ *but* $\hat{a}(t)| \varphi_1 \rangle =???$ *for* $t > 0$.

(iii) Now let's change the quantum system to a spin-1/2 with Hamiltonian

$$\hat{H} = \frac{\hbar\Omega}{2} \hat{\sigma}_x. \tag{8}$$

Show that the time evolution operator is

$$\hat{U}(t) = \begin{bmatrix} \cos(\Omega t/2) & -i\sin(\Omega t/2) \\ -i\sin(\Omega t/2) & \cos(\Omega t/2) \end{bmatrix} \tag{9}$$

Hint: Note that $\hat{\sigma}_x^2 = \mathbb{1}$, $\hat{\sigma}_x^3 = \hat{\sigma}_x$, $\hat{\sigma}_x^4 = \mathbb{1}$ *etc. Use the power series for* exp, sin, cos *and then plug it all together.*
Solution: These power series are $\exp(x) = \sum_{k=0}^{\infty} x^k/k!$, $\sin(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k+1}/(2k+1)!$, $\cos(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k}/(2k)!$. *We the start with*

$$\begin{aligned}
\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} &= \sum_{k=0}^{\infty} \frac{\left(-\frac{i}{\hbar}t\right)^k \hat{H}^k}{k!} = \sum_{k=0}^{\infty} \frac{\left(-\frac{i\Omega}{2}t\right)^k \hat{\sigma}_x^k}{k!} \\
&\overset{hint}{=} \sum_{k=0}^{\infty} \underbrace{(-i)^{2k}}_{=(-1)^k} \frac{\left(\frac{\Omega}{2}t\right)^{2k}}{(2k)!} \underbrace{\hat{\sigma}_x^{2k}}_{=\mathbb{1}} + \sum_{k=0}^{\infty} \underbrace{(-i)^{2k+1}}_{=(-i)(-1)^k} \frac{\left(\frac{\Omega}{2}t\right)^{2k+1}}{(2k+1)!} \underbrace{\hat{\sigma}_x^{2k+1}}_{=\hat{\sigma}_x} \\
&= \cos(\Omega t/2)\mathbb{1} - i\sin(\Omega t/2)\hat{\sigma}_x,
\end{aligned} \tag{10}$$

which has the matrix representation (**??**)*, as we wanted to show.*

(iv) From that, find the time evolving state from initial state $|\,\Psi(t=0)\,\rangle = |\downarrow\rangle$ in the Schrödinger picture and compare the probability $p_\uparrow$ to be in $|\uparrow\rangle$ with section 9.3.5.
Solution: We apply the time-evolution operator to the initial state

$$\hat{U}(t)|\,\Psi(t=0)\,\rangle = \begin{bmatrix} \cos(\Omega t/2) & -i\sin(\Omega t/2) \\ -i\sin(\Omega t/2) & \cos(\Omega t/2) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -i\sin(\Omega t/2) \\ \cos(\Omega t/2) \end{bmatrix} \tag{11}$$

The probability $p_\uparrow$ is found from the mod-square of the first component as $p_\uparrow = \sin^2(\Omega t/2)$, as in Eq. (9.53) for $\Delta = 0$ and hence $\Omega_{eff} = \Omega$.

(v) **Bonus:** To see this in the Heisenberg picture we define the projector $\hat{P}_\uparrow = |\uparrow\rangle\langle\uparrow|$ so that $p_\uparrow = \langle\,\hat{P}_\uparrow\,\rangle$. Then find $\langle\,\hat{P}_\uparrow(t)\,\rangle$, with $\hat{P}(t) = \hat{U}^\dagger(t)\hat{P}_\uparrow\hat{U}(t)$.
*Solution: We already know the time-evolution operator from (**??**), hence we can use Eq. (3.67):*

$$\begin{aligned}\hat{P}_{\uparrow,H}(t) = \hat{U}^\dagger(t)\hat{P}_\uparrow(0)\hat{U}(t) &= \begin{bmatrix} \cos(\Omega t/2) & i\sin(\Omega t/2) \\ i\sin(\Omega t/2) & \cos(\Omega t/2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\Omega t/2) & -i\sin(\Omega t/2) \\ -i\sin(\Omega t/2) & \cos(\Omega t/2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\Omega t/2) & i\sin(\Omega t/2) \\ i\sin(\Omega t/2) & \cos(\Omega t/2) \end{bmatrix} \begin{bmatrix} \cos(\Omega t/2) & -i\sin(\Omega t/2) \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos^2(\Omega t/2) & -i\cos(\Omega t/2)\sin(\Omega t/2) \\ i\cos(\Omega t/2)\sin(\Omega t/2) & \sin^2(\Omega t/2) \end{bmatrix} \end{aligned} \tag{12}$$

Taking the expectation value in the (eternal, time-independent) state $|\downarrow\rangle$ gives us $\sin^2(\Omega t/2)$ as before!

Stage 2 Interaction picture Consider the time dependent two level system (spin-1/2) that we had looked at in example 71 with Hamiltonian in matrix form

$$\underline{\underline{H}} = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}}_{=\hat{H}^{(0)}} + \underbrace{\begin{pmatrix} 0 & \kappa(t) \\ \kappa(t) & 0 \end{pmatrix}}_{=\hat{H}'(t)}. \tag{13}$$

with the same $\kappa(t)$ as used there.

(i) Find the time evolution of the operator $\hat{S}_x = \hbar(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|)$ in the interaction picture.
Solution: We require the free time evolution operator from Eq. (9.62) [setting $t_0 = 0$ for simplicity]

$$\hat{U}^{(0)}(t) = e^{-\frac{i}{\hbar}\hat{H}^{(0)}t} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i\Delta}{\hbar}t} \end{pmatrix} \tag{14}$$

*The reason why it is always easy to read of the operator-exponential for a diagonal operator is the same as discussed around Eq. (**??**). Now we can form*

$$\hat{S}_x(t) = \hat{U}^{(0)\dagger}(t)\hat{S}_x\hat{U}^{(0)}(t) = \hbar \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\Delta}{\hbar}t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i\Delta}{\hbar}t} \end{pmatrix}$$
$$= \hbar \begin{pmatrix} 0 & e^{-\frac{i\Delta}{\hbar}t} \\ e^{\frac{i\Delta}{\hbar}t} & 0 \end{pmatrix}. \tag{15}$$

(ii) Write the interaction picture state in the usual spin basis

$$|\,\Psi_I(t)\,\rangle = c_{I,\uparrow}(t)|\uparrow\rangle + c_{I,\downarrow}(t)|\downarrow\rangle, \tag{16}$$

and find the evolution equation for the time dependent coefficients
*Solution: We know this state evolves according to Eq. (9.68) for which we require $\hat{H}_I'(t)$, i.e. the interaction Hamiltonian from (**??**) but taken in the interaction picture:*

$$\hat{H}_I' = \hat{U}^{(0)\dagger}(t)\hat{H}'\hat{U}^{(0)}(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\Delta}{\hbar}t} \end{pmatrix} \begin{pmatrix} 0 & \kappa(t) \\ \kappa(t) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{i\Delta}{\hbar}t} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \kappa(t)e^{-\frac{i\Delta}{\hbar}t} \\ \kappa(t)e^{\frac{i\Delta}{\hbar}t} & 0 \end{pmatrix}. \tag{17}$$

Insertion into $i\hbar\frac{d}{dt}|\,\Psi_I(t)\,\rangle = \hat{H}_I'(t)|\,\Psi_I(t)\,\rangle$ and subsequent projection onto $\langle\uparrow|$ and $\langle\downarrow|$ gives:

$$\dot{c}_{I,\uparrow}(t) = \kappa(t)e^{-\frac{i\Delta}{\hbar}t}c_{I,\downarrow}(t),$$
$$\dot{c}_{I,\downarrow}(t) = \kappa(t)e^{\frac{i\Delta}{\hbar}t}c_{I,\uparrow}(t). \tag{18}$$

(iii) BONUS: Using both of the above, find the time evolution of the expectation value $\langle\,\hat{S}_x\,\rangle(t)$ in the interaction picture, starting from the initial state $|\,\Psi(t=0)\,\rangle = |\uparrow\rangle$ (might need mathematica).
Solution: To be provided later

Stage 3 Higher-order Time dependent perturbation theory

(i) How do we move to higher-order time dependent perturbation theory and what is the physical picture of the structure of quantum dynamics that it suggests to us?
Solution: See week 10 lecture notes.