PHY 304, II-Semester 2023/24, Tutorial 2

25. Jan. 2024

Discuss on your table in AIR. When all teams finished a stage, make sure all students at your table understand the solution and agree on one by using the board.

Stage 1 Symmetries For the following Hamiltonians, identify all symmetries, write the operators implementing the symmetry transformation and state the resulting conservation laws and degeneracies in the spectrum of the Hamiltonian.

(i) Free particle

$$\hat{H} = \frac{\hat{p}^2}{2m} \tag{1}$$

(ii) Mexican hat potential ($\alpha, \beta > 0$)

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} - \alpha(\hat{x}^2 + \hat{y}^2) + \beta(\hat{x}^2 + \hat{y}^2)^2 \tag{2}$$

(iii) Anisotropic 2D oscillator ($\omega_x \neq \omega_y$)

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2}m(\omega_x^2\hat{x}^2 + \omega_y^2\hat{y}^2) \tag{3}$$

Stage 2 Atom interacting with light: You have heard in PHY106 that light can cause transitions between electronic states of atoms, such as the $|\,\phi_{n\ell m}\,\rangle$ states of the Hydrogen atom in week 10. Let us brutally simplify the Hydrogen atom, and consider only two of its internal states, a ground state $|\,g\,\rangle$ and an excited state $|\,e\,\rangle$ and assume this atom is irradiated with laser light.

It turns out (in a preview of PHY402), that with some approximations, this atom can be described by an effective Hamiltonian

$$\hat{H} = \begin{bmatrix} 0 & \frac{\Omega}{2} \\ \frac{\Omega}{2} & -\Delta \end{bmatrix}. \tag{4}$$

We wrote a matrix representation wrt. states $|\,g\,\rangle$ and $|\,e\,\rangle$ (in this order). Ω is related to the laser intensity, and $\Delta = \omega_0 - \omega_{eg}$ to the difference between the light frequency ω_0 and the transition frequency $\omega_{eg} = |E_e - E_g|/\hbar$ (called a detuning).

We call the light "far off resonant" if $\Omega \ll |\Delta|$.

(i) Use this conditions to define a suitable splitting of the Hamiltonian to use perturbation theory, and then apply it to find the first and second order corrections to energies and first order correction to states.

(ii) If the light intensity varies with position $\Omega(\mathbf{r})$ discuss the resultant mechanical effect on the atom for all signs of Δ.

Stage 3 Perturbed particle in the box: Consider a charged particle in an infinite square well potential from 0 to a, subject to an external electric field. The latter causes a an addition $\hat{H}' = q\mathcal{E}_0(x - a/2)$ to the Hamiltonian.

(i) Using perturbation theory, find the first order correction to all energy levels.

(ii) Then write down the second order correction to the ground state energy, and without evaluating integrals discuss which is the dominant contribution [assuming $0 < \langle \phi_k | \hat{H}' | \phi_1 \rangle \sim 1/k$ for large k] and what sign the energy shift has. *Bonus: AFTER the tutorial you may confirm the above scaling of matrix elements by explicitly evaluating the integrals.*

(iii) Also write the first order correction to the wavefunction, again discuss which is the dominant contribution and what it qualitatively does to the ground-state of the particle. (Without evaluating the integrals, as above)

(iv) Revisit the online app http://www.falstad.com/qm1d/ and select “setup: infinite well + field”, particle mass = minimal and place the well width slider bar at about half the range, then apply a small field strength to check whether your conclusions for the previous two items were correct.