

PHY 304, II-Semester 2023/24, Tutorial 1 solution

Stage 1 Checklist Let's play a revision game for QM-I on your table: Everyone in turn quickly answer a question from below. Everyone then can agree, disagree or discuss. If you don't know the answer when it is your turn, pass or pick another question, but take note of those points for revision later.

Solution: The main point of the below was to get you to think about all the items and thus assist in identifying questions that you are uncomfortable with, or triggering discussion. As such there may be no real (unique) solution. Below I inserted what I would answer myself.

(a) **Basic structure of quantum mechanics** Instead of a well defined position and momentum as in classical mechanics, the quantum mechanical particle is described by a wave function $\Psi(x, t) \in \mathbb{C}$. Answer the following questions as a self-check: What is the interpretation/meaning of this wavefunction? *It captures all information about the physical state of the particle. $|\Psi(x_0, t)|^2 dx$ is the probability to find the particle at time $t = 0$ in the infinitesimal interval $[x_0, x_0 + dx)$. The phase of the wavefunction may carry information about a net motion of the particle.* *___*

Why is it complex and what does the complex part of the numbers tell us? *We know from experiment that the momentum of a quantum particle is defined by its de-Broglie wavelength $p = h/\lambda$ and its energy by its frequency $E = h\nu$. It is not possible to write a wave equation that provides the correct dispersion relation $E = p^2/(2m)$ without complex numbers. For what they mean see last sentence prior point.* *___*

What is the value of $\int dx |\Psi(x, t)|^2$ and why? *$\int dx |\Psi(x, t)|^2 = 1$ since the total probability to find the particle "anywhere" must be one.* *___*

What is the importance of a global complex phase $\Psi(x, t) \rightarrow e^{i\varphi} \Psi(x, t)$ with $\varphi \in \mathbb{R}$? *None. A global phase like this never matters. In contrast a relative phase between two basis functions, such as in $\phi_1(x) + e^{i\varphi} \phi_2(x)$ does matter.* *___*

Which equation governs how the wavefunction changes in time? *The TDSE.* *___*

What is the meaning of stationary state and how do we find those? *These are states for which the probability distribution does not change in time, we find them by solving the TISE.* *___*

(b) **Measurements in quantum mechanics** How do we find the probability of a measurement of a certain outcome for any observable? *We solve the eigenvalue problem $\hat{O}|\varphi_n\rangle = o_n|\varphi_n\rangle$. Then we expand the current state of the particle as $|\Psi\rangle = \sum_n c_n|\phi_n\rangle$. Then the probability to find o_k is $|c_k|^2$.* *___*

Why do we need operators in quantum mechanics? *We see in the generalized HUP that uncertainty relations only exist for non-commuting observables. Operators/matrices may not commute, in contrast simple*

functions always would commute. Since experiments show the existence of an uncertainty relation, we are forced to design quantum mechanics based on operators. ___.

What happens to the quantum state while we find a certain measurement result?___ In the example above, the state collapses into $|\varphi_k\rangle$. In more complicated cases, we can express this through a projection operator $\hat{P} = |\varphi_k\rangle\langle\varphi_k|$, using $|\Psi\rangle \rightarrow \hat{P}|\Psi\rangle$ and then renormalising the state to one. ___.

How do we find the mean of a large number of measurements?___ This is given by the expectation value $\langle\hat{O}\rangle$ ___.

How do we find the uncertainty or standard deviation of a large number of measurements?___ We use the formula $\Delta O = \sqrt{\langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2}$, based on the usual definition of standard deviation in statistics. ___.

(c) **Solutions of the TISE and why we need them** For what do we need the TISE? ___ If we want to know how a quantum state evolves in time. ___.

In which cases and how can we also find time-dependent information (time-evolution) from the TISE?___ When the Hamiltonian does not explicitly depend on time, we can find all the required information about time evolution from the energies E_n , energy eigenstates $|\phi_n\rangle$ and initial expansion coefficients $|\Psi(t=0)\rangle = \sum_n c_n(0)|\phi_n\rangle$ ___.

Which nice properties do solutions of the TISE have?___ They form an orthonormal basis of the Hilbertspace. The time evolution of a single solution (not a superposition) is trivial $\Psi(x,t) = e^{-iEt/\hbar}\phi_n(x)$ ___.

. List a few practical aspects needed to solve the TISE, how many different methods do you know?___ We solve it as a boundary value problem. So we need to know all boundary/connection condition and in the end normalise our solutions. We had seen methods based on (i) connecting trigonometric solutions in piecewise constant potentials, (ii) power series, (iii) algebraic solutions via commutators. ___.

What is the meaning of degeneracy?___ A degenerate eigenvalue is one that is shared by several non-equal eigenvectors/eigenfunctions. It implies that within the space spanned by those vectors, eigenvectors are not unique up to a phase. ___.

List some classifications of solutions of the TISE ___ There are bound-states and scattering states. ___.

List typical properties of bound state solutions of the TISE___ They have to vanish at infinity. Their energy eigenvalues are quantized. The higher excited the bound-state, the more nodes does the wavefunction have. ___.

(d) **Solutions of the TDSE and why we need them** Why do we need the TDSE in addition to the TISE?___ The TDSE is in some sense more fundamental (since we derived the TISE from it). But also we require it to know the time evolution if the Hamiltonian DOES depend on time ___.

How many methods to solve the TDSE do you know? ___ *On a computer, using a matrix representation and by expanding the state in terms of energy eigenstates and using TISE solutions* ___.

List a few physical phenomena for which knowing the TISE is not enough ___ *Absorption and emission of light, driven systems with a time varying external field.* ___.

- (e) **Uncertainty relations** What is the basic mathematical origin of uncertainty relations in quantum mechanics? List an intuitive and a formal reason. ___ *The origin is that particles are described by waves. A wave cannot be localized without losing knowledge of its wavelength. Formally, the origin are non-commuting operators that represent observables.* ___.

Which uncertainty relations do you know? ___ $\Delta x \Delta p > \hbar/2$, $\Delta E \Delta t > \hbar/2$, $\Delta L_x \Delta L_y > \hbar |\langle \hat{L}_z \rangle|/2$, same for spin. ___.

What is the relation between uncertainty relations and operators sharing eigenfunctions? ___ *Iff two operators share a complete set of eigenfunctions, they commute.* ___.

What is special with the energy-time uncertainty relation, and why do we have to be careful using it? ___ *Time is not an operator. Hence one has to pay extra attention to the definition of “characteristic energy uncertainty” and “characteristic timescale” involved in it.* ___.

- (f) **Quantum effects** List at least six quantum mechanical phenomena that are in an essential way different from behaviour in classical mechanics. ___ *quantum tunneling, quantum reflection, interference, quantisation of observables, measurements affect the state, outcome of measurements is probabilistic* ___.

- (g) **Algebraic solutions to quantum problems** With which trick can you often avoid finding all the eigenstates and energies from the TISE, and instead use the action of operators onto states directly? ___ *We can find the commutation relations of essential operators, and find then a large amount of information about the eigenstate structure just from those commutation relations. We did this for the harmonic oscillator in week 4, and for angular momentum, in week 9* ___.

- (h) **Angular momentum** How is angular momentum dealt with in quantum mechanics?

___ *With a vector operator the components of which satisfy the angular momentum commutation relations.* ___.

Which different types of angular momentum do you know?

___ *Orbital angular momentum, associated with motion about the origin and represented by angular derivatives of the 3D wavefunction in spherical polar coordinates, and intrinsic angular momentum or spin, as a property*

of fundamental particles. ___.

What rules to angular momentum quantum numbers fulfill?

___ $l = 0, 1, 2, 3, \dots$, $s = 0, 1/2, 1, 3/2, 2, \dots$, $-l \leq m_l \leq l$ ___.

Which links between angular momentum and particle motion do you know from classical mechanics and how are they preserved in quantum mechanics?

___ *Angular momentum is orthogonal to the plane of motion for a classical particle in a central potential. This is still “sort of true” in quantum mechanics, subject to satisfying the 3D uncertainty relations between position and momentum, which require us to make that plane “blurry”.* ___.

How do we add angular momenta of two particles? ___ *We define a total angular momentum operator $\hat{L}_{tot} = \hat{L}_1 + \hat{L}_2$ and then find $|\ell_1 - \ell_2| \leq \ell_{tot} \leq \ell_1 + \ell_2$ in integer steps* ___.

- (i) **Many-Particles or dimensions** What changes in the math when you move from a single particle to many particles? or from one dimension to many dimensions? ___ *The wavefunction gets an extra argument (dimension) for each spatial dimension. In 3D it then gets 3 extra dimensions per particle.* ___.

How can we often tackle those complications to resort back to our easier 1D solutions? ___ *We can use separation of variables to reduce higher dimensional problems to lower dimensional ones, or many particle problems to single particles ones. This requires dimensions to be uncoupled or particles to not interact.* ___.

Explain the concept of indistinguishable particles in quantum mechanics?

___ *If two particles have the exact same identity, the uncertainty relation prohibits us from assigning them separate identities, since we can never track them once they moved closer to each other than the spatial uncertainty.* ___.

What is the fundamental consequence of this principle? ___ *The wavefunction must be symmetric or anti-symmetric under exchange of two identical particles.* ___.

Which classification of particles does it lead to? ___ *Fermions and Bosons* ___.

What is entanglement and why is it interesting? ___ *An entangled two particle state is one that cannot be written as a product of two single particle states.* ___.

Stage 2 Questions and answers: A first step at having mastered some material is to be able to ask questions about it, and a second step is to be able to answer such questions. Perhaps guided by stage 1, I would like to ask each of you to make two large lists:

- (i) Topics from the QM-I course that you feel you understood well.

(ii) topics from the QM-I course that you feel you understood less well.

Share these lists within your group. Then I would like to ask all those which comfortable with a topic [i.e. it was on their list (i)] to explain it to those group-members that are not [i.e. have it on their list (ii)]. This will be beneficial for both sides: only when you can explain something have you fully understood it, and often you only understand it when you attempt to explain it. Afterwards, please make a group-wise collection again with (i) and (ii), in particular listing all topics that none of you felt happy to explain or all/most of you felt comfortable with. Sent this final list with your group ID and all student names to your TA.

Solution: None/your own

Stage 3 Entanglement: (*Only do this if you have done the revision part earlier, or are confident you need none*). For each of the following states of two spin-1/2 particles: (i) find the probability to measure the first spin to be “up”, (ii) find the state after measuring the second one in “up”, (iii) find the probability to measure first one in “up” after we have measured the second in “up”. Discuss.

$$(i) |\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$(ii) |\Psi\rangle = \frac{1}{2} (|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$

Hint: The outcome of a measurement can be found using projection operators \hat{P} , compare also postulates in section 3.6. If you measure the first particle to be in state $|\varphi_k\rangle$, you project the wavefunction before measurement $|\Psi\rangle$ according to $\hat{P}|\Psi\rangle$, and then re-adjust normalisation to one. In that expression

$$\hat{P} = |\varphi_k\rangle\langle\varphi_k| \otimes \mathbb{1} \otimes \mathbb{1}, \quad (1)$$

which acts as identity $\mathbb{1}$ onto all particles that are not the first.

Solution: (i) We the probability to measure the first spin to be “up” (without making a statement about the second spin), but summing up all probabilities for the first spin to be up, for all possible results of the second spin. For the first function thus $p = 1/2$. (ii) After measuring the second one in up, the projection gives us $|\Psi\rangle = |\downarrow\uparrow\rangle$. Subsequent to that measurement, the probability to measure first one in “up” is zero. We can write the second wavefunction as $|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$. From the previous form we can read off: (i) $p = 1/4 + 1/4 = 1/2$. From the second form it is easier to see that after the measurement $|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \otimes |\uparrow\rangle$. After that measurement we still have $p = 0$.

The first example is an entangled state, in which case the measurement on the second spin changes the probabilities of measurements on the first one. The second example is a separable state, where this is not the case.

Stage 4 Indistinguishable particles: (*Only do this if you have done the revision part earlier, or are confident you need none*). Which of the following is a valid quantum state for the two particles listed (\mathcal{N} normalises the wavefunction).

(i) Two ^{87}Rb atoms: $\Psi(x_1, x_2) = \mathcal{N}e^{-\frac{x_1^2+x_2^2}{2\sigma^2}}$.

(ii) Two ^{40}K atoms: $\Psi(x_1, x_2) = \mathcal{N}e^{-\frac{x_1^2+x_2^2}{2\sigma^2}}$.

(iii) A Helium atom and a Hydrogen atom: $\Psi(x_1, x_2) = \mathcal{N}e^{-\frac{x_1^2}{2\sigma^2}}e^{ikx_2}$.

Solution: Rubidium has $Z = 37$ protons so the isotope ^{87}Rb has $N = 50$ neutrons, i.e. an even number. That makes it a Boson, hence the wavefunction must be symmetric under the swap $x_1 \leftrightarrow x_2$, which it is, so this is a valid quantum states for two identical Bosons. Potassium has $Z = 19$ protons so the isotope ^{40}K has $N = 21$ neutrons, i.e. an odd number. That makes it a Fermion, hence the wavefunction must be anti-symmetric under the swap $x_1 \leftrightarrow x_2$, which it is not, so this is an invalid quantum states for two identical Fermions. The last example are two different particles, so there are no special symmetry requirements on the wavefunction and the state give is valid.