Week 9

PHY 304 Quantum Mechanics-II
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These notes are provided for the students of the class above only. There is no guarantee for correctness, please contact me if you spot a mistake.

9.2.2 Adiabatic change of the Hamiltonian

Further reading: For this section (Proof of adiabatic theorem), please also refer to Sakurai, section 5.6. (SA), or Griffith 2nd edition only.

The opposite of the sudden change of $\hat{H}$ in (9.1.2) is a very slow change. Suppose we separately diagonalize the Hamiltonian at each moment in time

$$\hat{H}(t)|\,\phi_n(t)\,\rangle = E_n(t)|\,\phi_n(t)\,\rangle. \tag{9.27}$$

Since $\hat{H}(t)$ depends on time, of course so do the instantaneous eigenstates and energies.

Now, if $\hat{H}(t)$ varies slow enough[22] we can show the

Quantum adiabatic theorem If the quantum system is initialized in a state $|\,\phi_n(t_i)\,\rangle$ at the initial time t_i, if $\hat{H}(t)$ varies slow enough between t_i and t_f, the quantum state n will be adiabatically followed such that it remains in $|\,\phi_n(t_f)\,\rangle$ at the final time.

- We can in fact make a stronger statement regarding superpositions and quantum phases, during the proof, where we also see what "slow enough" means.
- Since we are referring to instantaneous eigenstates of the Hamiltonian, in general $|\,\phi_n(t_f)\,\rangle \neq |\,\phi_n(t_i)\,\rangle$ as long as $\hat{H}(t_f) \neq \hat{H}(t_i)$, in contrast to the previous section.

Proof: The eigenstates in (9.27) for any time t form a basis of the Hilbertspace hence we can write

$$|\,\Psi(t)\,\rangle = \sum_n d_n(t)|\,\phi_n(t)\,\rangle = \sum_n c_n(t)e^{-i\theta_n(t)}|\,\phi_n(t)\,\rangle. \tag{9.28}$$

[22] to be defined shortly

with $\theta_n(t) = -\int_0^t dt'\, E_n(t')/\hbar$. In the second equality we simply redefined our coefficients in a way that will be useful later. We now plug (9.28) into the TDSE (3.8) to find

$$\begin{aligned} i\hbar \sum_n \left[\dot{c}_n(t) e^{-i\theta_n(t)} |\, \phi_n(t)\,\rangle - \frac{i}{\hbar} c_n(t) E_n(t) e^{-i\theta_n(t)} |\, \phi_n(t)\,\rangle + c_n(t) e^{-i\theta_n(t)} \frac{\partial}{\partial t} |\, \phi_n(t)\,\rangle \right] \\ = \sum_n c_n(t) e^{-i\theta_n(t)} \underbrace{\hat{H}(t) |\, \phi_n(t)\,\rangle}_{=E_n(t)|\,\phi_n(t)\,\rangle}, \end{aligned} \tag{9.29}$$

and then project onto $\langle\, \phi_k(t)\,|$ to tidy up the sum. We can also divide by $e^{-i\theta_k(t)}$, then

$$i\hbar \dot{c}_k(t) + c_k(t) E_k(t) + i\hbar \sum_n c_n(t) e^{i[\theta_k(t) - \theta_n(t)]} \langle\, \phi_k(t)\,| \frac{\partial}{\partial t} |\, \phi_n(t)\,\rangle = c_k(t) E_k(t), \tag{9.30}$$

and thus

$$\dot{c}_k(t) = -\sum_n c_n(t) e^{i[\theta_k(t) - \theta_n(t)]} \langle\, \phi_k(t)\,| \frac{\partial}{\partial t} |\, \phi_n(t)\,\rangle. \tag{9.31}$$

Now the scalar product here is a bit tricky and in the general case we don't yet know what it will give. For a time-independent Hamiltonian you can convince yourself that the line (9.31) reduces to Eq. (1.69), though, so all is consisten. For a time-dependent one, we need one more step, starting with taking the time derivative of Eq. (9.27), which gives us

$$\frac{\partial}{\partial t} \hat{H}(t) |\, \phi_n(t)\,\rangle + \hat{H}(t) \frac{\partial}{\partial t} |\, \phi_n(t)\,\rangle = \frac{\partial}{\partial t} E_n(t) |\, \phi_n(t)\,\rangle + E_n(t) \frac{\partial}{\partial t} |\, \phi_n(t)\,\rangle. \tag{9.32}$$

Taking the scalar product with $\langle\, \phi_k(t)\,|$ assuming $k \neq n$

$$\langle\, \phi_k(t)\,| \frac{\partial}{\partial t} \hat{H}(t) |\, \phi_n(t)\,\rangle + \underbrace{\langle\, \phi_k(t)\,| \hat{H}(t)}_{=\langle\, \phi_k(t)\,| E_k(t)} \frac{\partial}{\partial t} |\, \phi_n(t)\,\rangle = \frac{\partial}{\partial t} E_n(t) \underbrace{\langle\, \phi_k(t)\,|\, \phi_n(t)\,\rangle}_{=0} + E_n(t) \langle\, \phi_k(t)\,| \frac{\partial}{\partial t} |\, \phi_n(t)\,\rangle, \tag{9.33}$$

which we can reform into the term we seek (for $k \neq n$)

$$\langle\, \phi_k(t)\,| \frac{\partial}{\partial t} |\, \phi_n(t)\,\rangle = \frac{\langle\, \phi_k(t)\,| \frac{\partial}{\partial t} \hat{H}(t) |\, \phi_n(t)\,\rangle}{E_n(t) - E_k(t)}. \tag{9.34}$$

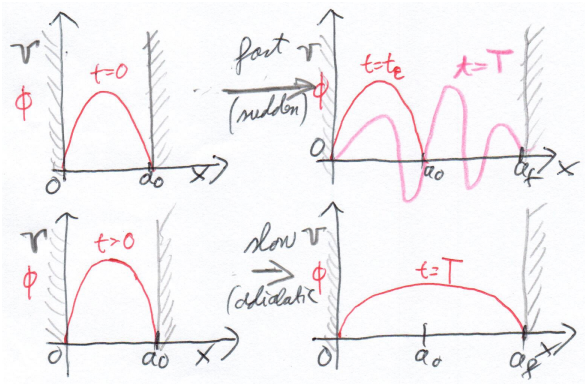
Insertion into (9.31) gives us:

$$\dot{c}_k(t) = -c_k(t) \langle\, \phi_k(t)\,| \frac{\partial}{\partial t} |\, \phi_k(t)\,\rangle - \sum_n c_n(t) e^{i[\theta_k(t) - \theta_n(t)]} \frac{\langle\, \phi_k(t)\,| \frac{\partial}{\partial t} \hat{H}(t) |\, \phi_n(t)\,\rangle}{E_n(t) - E_k(t)}, \tag{9.35}$$

where we split up the sum into the term with $n = k$ and all others. We see that the first term can never move quantum amplitude from one state n to another state k. I.e. if we start with $|c_k|^2 = \delta_{kj}$ for some j, this property will not change in time, if only the first term is nonzero. All transitions between quantum states must thus be from the second term. Thus we now define "$\hat{H}(t)$ varies slow enough" for the quantum adiabatic theorem to apply, as "slow enough for the second term to be negligible. Unfortunately that term is rather complex so stating mathematically neatly when it will be negligible is difficult. However we can clearly see a few things:

- The slower the variation of $\hat{H}(t)$, the smaller will be its derivative $\frac{\partial}{\partial t}\hat{H}(t)$, thus the smaller the entire term. Hence the requirement for slow variations is visible.
- This must be compared with the denominator, which contains all energy differences $\Delta E_{nm}(t) = E_n(t) - E_k(t)$ between instantaneous eigenstates. Thus the higher those energy differences, the faster can we afford the Hamiltonian to vary without causing a transition.
- Any more specific conditions on what constitutes "slow enough" depend on the problem at hand, see example 73.

Example 72, Square well potential with moving wall: Let us first see in an example what section 9.1.2 versus section 9.1.3 say:



left: Consider the infinite square well potential (2.10), where the size of the well $a(t)$ is time dependent. Initially, for $t < 0$ let it be a_0 and the particle in the ground state $\phi_1(x)$

$$\Psi(x, t < 0) = \sqrt{\frac{2}{a_0}} \sin\left(\frac{\pi x}{a_0}\right), \qquad (9.36)$$

for that width a_0. Now, if we suddenly change $a(t > 0) = a_{\text{fin}} > a_0$, the discussion in section 9.1.2 tells us that immediately after the change, we still have $\Psi(x, t_\epsilon) = \phi_1(x) = \sqrt{2/a_0}\sin(\pi x/a_0)$ as shown.

Subsequently this state will evolve in time, since $\Psi(x, t_\epsilon)$ is not an eigenstate of the box of width a_{fin}. We can find $\Psi(x, t)$ e.g. using Eq. (1.70).
In contrast, if we vary $a(t)$ slow enough to $a(T) = a_{\text{fin}}$ with large T, the discussion in section 9.1.3 tells us that the system will arrive in the ground-state of this new box, $\Psi(x, T) = \sqrt{2/a_{\text{fin}}}\sin(\pi x/a_{\text{fin}})$.

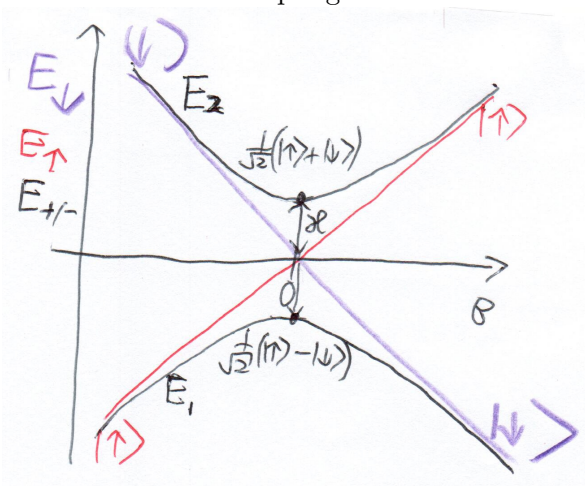
Example 73, Landau Zener transition: Consider an electron in an external magnetic field $\mathbf{B}(t) = B(t)\mathbf{e}_z$. Due to the spin's magnetic moment, the energy can be found in Eq. (7.83). Let us further assume there is some interaction κ that can flip the spin, such that the Hamiltonian becomes

$$\hat{H} = \frac{\Delta E}{2}|\uparrow\rangle\langle\uparrow| - \frac{\Delta E}{2}|\downarrow\rangle\langle\downarrow| + \kappa\big(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\big) \qquad (9.37)$$

with $\Delta E(t) = 2\mu_B B(t)$ or in matrix form

$$\underline{\underline{H}} = \begin{pmatrix} \frac{\Delta E(t)}{2} & \kappa \\ \kappa & -\frac{\Delta E(t)}{2} \end{pmatrix}. \qquad (9.38)$$

Example continued: The diagram below shows the eigen-energies of the Hamiltonian, with and without coupling κ.



left: Let us assume we start at $B = B_0 < 0$ (magnetic field anti-parallel to $\mathbf{e}_z$) with the electron in state $|\uparrow\rangle$ and now slowly change $B(t) = B_0 + \Gamma_B t$ such that it is positive in the end. Through the magnetic field the instantaneous eigenstates $|\,\phi_n(t)\,\rangle$ of (9.27) and energies $E_n(t)$ also depend on time with energies drawn in the figure. Initially the electron thus is in $|\,\phi_1\,\rangle = |\uparrow\rangle$. After the magnetic field reached large positive values, what spin state will the electron be in?

We can answer two extreme cases based on our discussion so far of section 9.2.
(a) If we vary the magnetic field very fast (instantaneous), the spin state does not change. Since we start in $|\uparrow\rangle$, we end in $|\uparrow\rangle$. However, for large B it is the second, higher, eigenstates that corresponds to this $|\,\phi_2\,\rangle \overset{B\to\infty}{\to} |\uparrow\rangle$. In terms of instantaneous eigenstates of the Hamiltonian, the electron thus changed eigenstate.
(b) If we vary B infinitely slowly, the adiabatic theorem guarantees that we remain in the instantaneous eigenstate that we started in $|\,\phi_1\,\rangle$, which however becomes $|\downarrow\rangle$ for large B. Here we thus did not change eigenstate, but since the eigenstate itself has changed, we changed the spin-state.
(c) In between, for finite rate of change of the Hamiltonian, based on Eq. (9.35) we expect some probability P to transfer from $|\,\phi_1\,\rangle$ to $|\,\phi_2\,\rangle$. The exact calculation is too technical to present it here, but for small P, it can be found as

$$P = e^{-\pi \frac{\kappa^2}{\mu_B \Gamma_B}} \tag{9.39}$$

(9.39) is the approximate Landau Zener transition probability and an extremely useful formula. The reason is a more general result, that eigenvalues of a Hamiltonian as a function of a single varying parameter "do not like to cross" (keyword: Wigner non-crossing theorem). As a result, even a more complicated multi-level spectrum of eigenvalues of $\hat{H}(B)$ as a function of a single parameter B has lots of so called avoided crossings of eigenvalues, which look like the figure above when zoomed into. For each of these avoided crossings, we can then use formula (9.39) when varying the parameter B as a function of time. In the more general context, we write

$$P = e^{-2\pi \frac{\kappa^2}{\hbar|\alpha|}} \tag{9.40}$$

where we removed the explicit reference to spins in a magnetic field, and just assume the energy difference between the two levels varies as $\Delta E = \alpha t$.

The above discussion has further implications for Molecular physics (PHY402), where we get a time-

dependence of the Hamiltonian for electrons, due to motion of the nuclei $\mathbf{R}_k(t)$. There are also interesting features arising through the structure of the first term in (9.35), related to the structure of Hilbertspace, which you can google/read up under the name "Berry[23] phase" or "geometric phase".

9.3 Light-matter interaction

We now finally will provide the examples promised for the results of section (9.1.4). The most obvious scenario in which we will see a periodic perturbation of a quantum system, is when we are hitting it with waves/radiation as the perturbation. Of these, the by far most important ones are of course electro-magnetic waves, which we thus first review briefly. We will then look at their interaction with a Hydrogen atom, but the results we find will have much wider implications for the interaction of any matter with radiation.

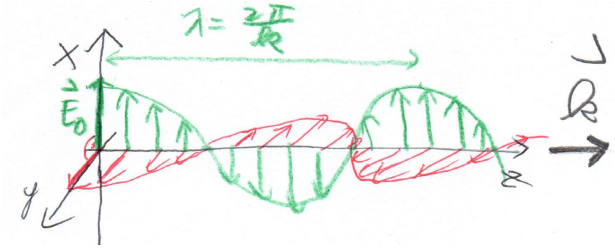
9.3.1 Electromagnetic waves

Recall that a monochromatic, linearly polarized plane electro-magnetic wave travelling in the positive z direction consists of orthogonal electric and magnetic fields

$$\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t), \tag{9.41}$$

$$\mathbf{B} = \frac{1}{c}\mathbf{k}\times\mathbf{E} \tag{9.42}$$

with $\mathbf{k} = k\mathbf{e}_z$ and $\mathbf{E}_0$ the polarisation vector of the wave ($\mathbf{E}_0 = E_0\mathbf{e}_x$ in the drawing below).



left: Electro-magnetic plane wave travelling in the positive z-direction. The electric field (green) is orthogonal to the magnetic field (red) and the propagation direction (black) at all times, and harmonically oscillates in space and time.

We know that a charge q at location $\mathbf{r}$ has an electrostatic potential energy $E_{\text{coul}} = q\mathbf{r}\cdot\mathbf{E}$ in an electric field $\mathbf{E}$, hence we take the

Hamiltonian for an electron in an electro-magnetic wave of frequency ω and wavenumber $\mathbf{k}$

$$\hat{H}_{elm} = -e\mathbf{r}\cdot\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{r} - \omega t). \tag{9.43}$$

- We have implicitly assumed that the motion of the electron is and remains non-relativistic, so that we can neglect the contribution due to the magnetic field. A more comprehensive general approach to charged particles in elm field will be presented in week 11.

[23] This is the most widespread nametag. The concept appears in so many disciplines that we could also call it "Pancharatnam-Longuet-Higgins-Berry" phase, and maybe more names I don't know about.

- Our main focus in the following will be on the interaction of visible light with Hydrogen atoms. In that case the light wavelength λ 500 nm is a lot larger than the size of the atom $\sim a_0 = 5\times10^{-11}$ m. Thus $\mathbf{k}\cdot\mathbf{r}\approx 0$ in (9.44) and we can write the Hamiltonian for an electron in an electro-magnetic wave in the dipole approximation as

$$\hat{H}_{elm} = \underbrace{\hat{\mathbf{d}}\cdot\mathbf{E}_0}_{=\hat{V}}\cos(\omega t). \tag{9.44}$$

  where $\hat{\mathbf{d}} = -q\hat{\mathbf{r}}$ is the dipole operator of the atom, compare example 50.

- Note that the dipole approximation is much less good for example when x-rays impinge on an atom, or visible light impinges on an electron in a very wide quantum dot. More generally, we can write a complex form for the cosine in (9.44) and expand $e^{i\mathbf{k}\cdot\mathbf{r}} \approx 1+i(\mathbf{k}\cdot\mathbf{r})-(\mathbf{k}\cdot\mathbf{r})^2/2+\cdots$, amounting to a multi-pole expansion of the electronic charge distribution.

- Now (9.44) takes the form (9.20) of a periodic perturbation, hence we can use Eq. (9.24) in the following.

9.3.2 Absorption, stimulated emission and spontaneous emission

Let us consider only an initial state $|\,i\,\rangle = |\,\phi_i^{(0)}\,\rangle = |\,n\ell m\,\rangle$ and a final state $|\,f\,\rangle = |\,\phi_f^{(0)}\,\rangle = |\,n'\ell'm'\,\rangle$ of the Hydrogen atom, taken from Eq. (4.91). For the application of Eq. (9.24) we require

$$V_{fi} = \langle\, f\,|\hat{V}|\,i\,\rangle = \langle\, n'\ell'm'\,|\hat{\mathbf{d}}\cdot\mathbf{E}_0|\,n\ell m\,\rangle = -\langle\, n'\ell'm'\,|ezE_0|\,n\ell m\,\rangle = -\mathfrak{p}E_0, \tag{9.45}$$

where we have assumed $t\mathbf{E}_0 = E_0\mathbf{e}_z$. In the last equality we have defined the transition dipole $\mathfrak{p} = e\langle\, f\,|z|\,i\,\rangle$ of the $i\to f$ transition (which is a complex number), which we can calculate from Eq. (4.91) with a 3D integration. We had seen in example 50 that $\mathfrak{p}=0$ unless $|\ell-\ell'|=1$ (we will get back to that shortly).

Using (9.45) we now find the

Atomic transition probability $|\,i\,\rangle\to|\,f\,\rangle$ due to monochromatic radiation with freq. ω

$$P_{if}(t) = \frac{|\mathfrak{p}E_0|^2}{\hbar^2}\frac{\sin^2\left[(\omega_{fi}-\omega)\frac{t}{2}\right]}{|\omega_{fi}-\omega|^2}. \tag{9.46}$$

- Not much happened compared to (9.45), except that we have now an interpretation of what causes the perturbation and what the states are. We have now finally seen what you have been told since PHY106: That radiating an atom with light at the transition frequency $\omega_{fi} = \hbar|E_f - E_i|$ can cause it to change its electronic state from $|\,i\,\rangle$ to $|\,f\,\rangle$ with some probability.

- Importantly note that this formula works exactly the same, regardless of which state hat the higher energy, thus compared to section 9.1.4 we have inserted a modulus in our definition of ω_{fi}. Justification: If we redid the calculation of section 9.1.4 assuming $\omega_{fi}<0$, the argument above Eq. (9.22) would require us to pick the second term in Eq. (9.21) instead. Following from there, we reach the same result, with the sign of ω_{fi} flipped.

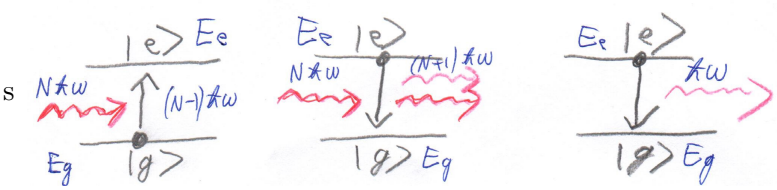
- Radiation causing an upwards transition in energy is called absorption and the downwards transition stimulated emission. We just learnt that both have the exact same probability.

- Note that we used a classical electromagnetic field where we should be using a quantized electromagnetic field, in what is known as quantum electro dynamics (QED). That would build in the fact that the field is made of discrete quanta called photons, each carrying an energy $E_{\text{phot}} = \hbar\omega$. In QED we then learn that the fundamental process underlying absorption and stimulated emission found above in first order perturbation theory is the removal or creation of exactly one photon in the radiation field. We now can easily understand the dependence on $\omega_{fi} - \omega$ of Eq. (9.46) by inspecting the figure below Eq. (9.45) (right) at large times: We require $\omega_{fi} - \omega = 0$, which is just due to energy conservation: The energy of the initial state and final state must be equal, see diagrams below.

- For short times, there is NO strict energy conservation, in agreement with the energy time HUP Eq. (3.56).

- A further feature that emerges in QED, is that there also is a probability for the atom to make the downwards transition even if there is no photon. This is called spontaneous emission. You can think of this as being "stimulated by vacuum fluctuations" of the elm field. These make sure that even if the elm field is in its ground-state, electric and magnetic field uncertainties are non-zero, in the same way that position and momentum uncertainties are non-zero in the ground-state of the simple harmonic oscillator.

We summarize the above discussion in the drawing below, that you had already seen in PHY106, but now we at last did at least some calculations to justify them.



left: (left) Absorption, (middle) stimulated emission and (right) spontaneous emission of photons by an atom. In each case the energy of the initial and final state (including the photon energies) is equal.

9.3.3 Incoherent transitions and Einstein argument (Bonus)

Turns out Einstein found a way to see that spontaneous emission must also be possible, even without knowing about QED. To appreciate this argument we would have to briefly consider incoherent transitions (arising due to a mix of frequencies ω in the elm field, instead of a monochromatic plane wave). One then looks at an atom in thermal equilibrium with a radiation field. You will see this in detail in PHY402, but if you want can already read and understand now the corresponding section in Griffith (3rd ed) 11.2.3. and 11.3.

9.3.4 Selection Rules

Above we had mainly discussed frequency and time dependence of $P_{if}(t)$. Another crucial ingredient of $P_{if}(t)$ is the matrix element $V_{fi} = -\langle\, n'\ell'm' \,|ezE_0|\, n\ell m \,\rangle$. This justifies our argumentation in example 50, that this matrix element is crucial for inter-atomic transitions.

We can evaluate this as a 3D integral and from that find the

Dipole selection rules An atomic transition $|\, n\ell m \,\rangle \to |\, n'\ell'm' \,\rangle$ due to a linearly polarised elm. field is only likely if

$$|\ell - \ell'| = 1, \qquad m = m'. \tag{9.47}$$

- The matrix element does not generate selection rules on n, but those are required by the need to be (near) resonance, since energies E_n in Hydrogen depend on n.
- These are called dipole selection rules because they follow from the matrix elements in Eq. (9.45) which contains the dipole operator of the atom. The pre-cursor, matrix elements in (9.44), contained $e^{i\mathbf{k}\cdot\mathbf{r}}$ instead which includes also all higher order electro-static multipoles. It turns out the higher orders have different selection rules (e.g. quadrupole $|\ell - \ell'| = 2$ but transition due to them are much weaker as long as $e^{i\mathbf{k}\cdot\mathbf{r}} \approx 1$ across the atom. Nonetheless one can thus induce these so called dipole forbidden transitions with very high intensity light.
- You will learn a lot more about selection rules in PHY402, such as interplay with light polarison and spin or total angular momentum states.

9.3.5 Two level system

In section 9.3.2 we had seen that emission and absorption of photons is much more likely for a resonant transition ($\omega_{ba} = (E_b - E_a)/\hbar = \omega$) than for other frequencies. Thus when a laser (which has a very well defined ω) interacts with atoms, it is often enough to only consider some two states in the Hilbert-space from the outset, which are resonant or near resonant with the laser, let us call these $|\, a \,\rangle$ and $|\, b \,\rangle$. We can then actually solve the complete TDSE, without resorting to perturbation theory. Let us write the time-dependent state as

$$|\Psi(t)\rangle = c_a(t)e^{-\frac{i}{\hbar}E_a t}|a\rangle - ic_b(t)e^{-i\Delta t}e^{-\frac{i}{\hbar}E_b t}|b\rangle, \tag{9.48}$$

where we have used the detuning $\Delta = \omega - \omega_{ba}$ and the special choice of $c_b(t)$ is a trick to simplify the problem as seen shortly.

The Hamiltonian is $\hat{H} = \hat{H}_0 + \hat{H}'(t)$ with Hydrogen Hamiltonian $\hat{H}_0$ from Eq. (4.71) and atom-light coupling from Eq. (9.20). Taking the matrix form (3.13) with only two states yields:

$$\underline{\underline{H}} = \begin{pmatrix} E_a & V_{ba}\cos(\omega t) \\ V_{ab}\cos(\omega t) & E_b \end{pmatrix}. \tag{9.49}$$

Now writing the TDSE (3.31) using (9.49) and the Ansatz (9.48), you can show in some few steps that:

$$i\hbar \begin{pmatrix} \dot{c}_a(t) \\ \dot{c}_b(t) \end{pmatrix} = \begin{pmatrix} 0 & V_{ba} \underbrace{e^{-i\omega t} \cos(\omega t)}_{=\frac{1}{2}(1+e^{-2i\omega t})} \\ V_{ba} e^{-i\omega t} \cos(\omega t) & -\hbar\Delta \end{pmatrix} \begin{pmatrix} c_a(t) \\ c_b(t) \end{pmatrix} \approx \underbrace{\begin{pmatrix} 0 & \frac{\hbar\Omega}{2} \\ \frac{\hbar\Omega}{2} & -\hbar\Delta \end{pmatrix}}_{=\hat{H}_{\text{eff}}} \begin{pmatrix} c_a(t) \\ c_b(t) \end{pmatrix} \tag{9.50}$$

where we have defined the Rabi frequency $\hbar\Omega = V_{ba}$.

The approximation $\approx$ is justified because the complex number $e^{-2i\omega t}$ will be very rapidly rotating in the complex plane and thus averages to zero for cases of interest. This is hence called rotating wave approximation. We see that the problem in the end can be described by an effective Hamiltonian $\hat{H}_{\text{eff}}$ which has become time-independent, so that we can solve the TISE with standard methods, e.g. (1.70).

Defining $\Omega_{\text{eff}} = \sqrt{\Omega^2 + \Delta^2}$, the solution for an initial state $|\,\Psi(0)\,\rangle = |\,a\,\rangle$ is

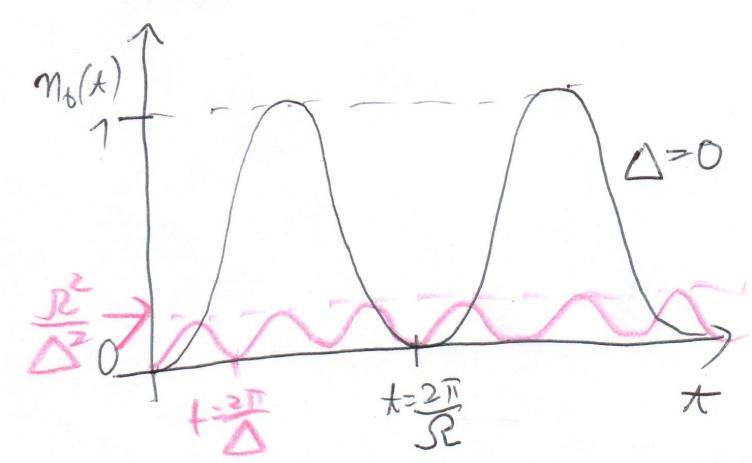
$$\begin{aligned} c_a(t) &= e^{i\frac{\Delta t}{2}} \left\{ \cos\left(\frac{t\Omega_{\text{eff}}}{2}\right) - i\frac{\Delta}{\Omega_{\text{eff}}} \sin\left(\frac{t\Omega_{\text{eff}}}{2}\right) \right\}. \\ c_b(t) &= -i\frac{\Omega}{\Omega_{\text{eff}}} e^{i\frac{\Delta t}{2}} \sin\left(\frac{t\Omega_{\text{eff}}}{2}\right). \end{aligned} \tag{9.51}$$

These amplitudes gives rise to the ubiquitous

Rabi oscillations of the population in a two-level system

$$n_b(t) = |\tilde{c}_b(t)|^2 = \frac{\Omega^2}{\Omega_{\text{eff}}^2} \sin^2\left(\frac{\Omega_{\text{eff}}}{2} t\right) \tag{9.52}$$

which are drawn below (we also know $n_a(t) = 1 - n_b(t)$ from normalisation).



left: Resonant Rabi-oscillations ($\Delta = 0$, black) versus detuned Rabi-oscillations ($|\Delta| \gg |\Omega|$, magenta, see example 74). We now know why we had called the matrix-element $V_{ab}/\hbar$ Rabi-frequency. It sets the frequency of population oscillations in our two level system, and is not to be confused with the frequency of the perturbation ω, which only enters the detuning Δ, but NOT the Rabi-frequency.

- For large detuning $\Delta \gg \Omega$, the probability to reach the state $|\,\phi_b\,\rangle$ remains small: $n_b(t)\big|_{\text{max}} =$

$\Omega^2/\Omega_{\text{eff}}^2 \approx (\Omega/\Delta)^2 \ll 1$. This often in retrospect justifies our approximation to consider only two atomic states, if the detuning to any other state would have been way larger than Δ.

Example 74, Detuned Rabi oscillations using perturbation theory: In the far-detuned limit $|\Delta| \gg |\Omega|$ you can convince yourself that it is justified to treat Eq. (9.49) using time-dependent perturbation theory, applying (9.24). For the full solution we found above we this limit amounts to $\Omega_{\text{eff}} \to |\Delta|$ and hence (9.3.5) becomes

$$n_b(t) \approx \frac{\Omega^2}{\Delta^2} \sin^2\left(\frac{\Delta}{2}t\right) \tag{9.53}$$

To apply instead (9.24) we read off from Eq. (9.49) that we have to set $V_{fi} = \hbar\Omega$ and $|\omega_{fi} - \omega|^2 = \Delta$, thus we see that time-dependent perturbation theory accurately captures the true solution in this limit.

Effective two level systems If you look carefully, you will see that the discussion in this section is not really special to a two-level atom. We could have any two quantum states of an unperturbed Hamiltonian, and any periodic perturbation. All the same calculation will be possible.
Hence, while we started with two atomic states coupled by laser light, Rabi oscillations are much more general and arise in any effective quantum two level system, for which the Hamiltonian in the space $\{|a\rangle, |b\rangle\}$ can be brought into the form of $\hat{H}_{\text{eff}}$, and any other states are even (much) farther detuned.