Week 11

PHY 304 Quantum Mechanics-II

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10 Electro-magnetic interactions

So far we have always described magnetic effects somewhat ad-hoc, such as when we coupled the magnetic moment due to a spin to a magnetic field as in (7.52). We shall now consider a more fundamental way to treat the effect of electro-magnetic fields in quantum mechanics which will also enable us to look at something we have avoided so far: The Lorentz force on a charged particle.

10.1 Gauge invariance

You know from classical physics that the Lorentz force is not conservative, i.e. it cannot be written as the gradient of a potential. Nonetheless we can write a Hamiltonian

Hamiltonian for a charged particle in an electromagnetic field

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}})\right)^2 + q\varphi(\hat{\mathbf{r}}). \tag{10.1}$$

from which the Hamilton equation provide the correct Lorentz force.

- Here we have followed our usual quantisation procedure and just replaced position and momentum in the classical Hamiltonian with operators. The particle has charge q and we can find the magnetic field as $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}(\hat{\mathbf{r}})$ from the vector potential and the electric field as $\mathbf{E} = -\boldsymbol{\nabla}\varphi(\hat{\mathbf{r}}) - \frac{\partial}{\partial t}\mathbf{A}(\hat{\mathbf{r}})$ from both, including the scalar potential.
- We had already used this without much discussion in Eq. (7.80).
- Inserting this into the TDSE gives us

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - q\mathbf{A}(\hat{\mathbf{r}},t)\right)^2 + q\varphi(\hat{\mathbf{r}},t)\right]\Psi(\mathbf{r},t) \tag{10.2}$$

- We know that the classical electromagnetic potentials are Gauge invariant, that means they themselves are not unique, since we can always replace them by

$$\varphi(\hat{\mathbf{r}},t)' = \varphi(\hat{\mathbf{r}},t) - \frac{\partial}{\partial t}\Lambda(\hat{\mathbf{r}},t), \qquad \mathbf{A}(\hat{\mathbf{r}},t)' = \mathbf{A}(\hat{\mathbf{r}},t) + \boldsymbol{\nabla}\Lambda(\hat{\mathbf{r}},t), \tag{10.3}$$

where $\Lambda(\hat{\mathbf{r}},t)$ is an arbitrary real function of space and time. You can check by insertion that the primed potentials will give you exactly the same electric and magnetic fields as the unprimed ones.

- Note however, that the TISE (10.2) directly contains the potentials (in contrast to Newton's equations, which only contain forces based on the fields). We can however show the

Gauge invariance of Schrödinger's equation If $\Psi(\mathbf{r},t)$ solves (10.2), then

$$\Psi(\mathbf{r},t)' = e^{iq\Lambda(\hat{\mathbf{r}},t)/\hbar}\Psi(\mathbf{r},t) \tag{10.4}$$

solves it with electromagnetic potentials being replaced as in (10.3).

- This is a local phase transformation. Thus the position probability distribution $|\Psi(\mathbf{r},t)'|^2 = |\Psi(\mathbf{r},t)|^2$ remains unchanged. It would appear that the momentum properties DO change, since e.g. applying $\hat{p} = -i\hbar\partial/\partial x$ to both sides yields an extra term from the product rule, but we have to be careful that for a Hamiltonian Eq. (10.1) not $\hat{\mathbf{p}}$ is the mechanical momentum, but instead we have to use the canonical momentum

$$\hat{\mathbf{p}}_c = \hat{\mathbf{p}} + q\mathbf{A}. \tag{10.5}$$

We know this from classical mechanics, but also for example can show from Eq. (10.2) the Ehrenfest theorem

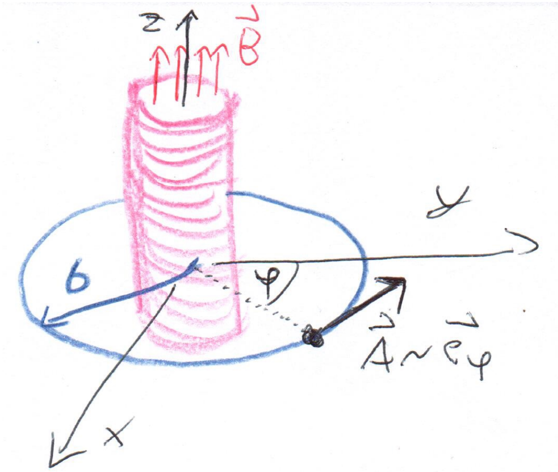
$$\frac{\partial}{\partial t}\langle\hat{r}\rangle = \frac{1}{m}\langle\hat{\mathbf{p}} + q\mathbf{A}\rangle. \tag{10.6}$$

- In fact one can turn the argument above around and ask the following: We know from basic quantum physics that a "global phase" of a quantum state does not matter (see QM-I section 1.6.1). What about "local phase"? By requiring the TDSE be invariant under the local phase transformation (10.4) ($U(1)$ symmetry), we see that it has to involve Gauge fields $\mathbf{A}(\hat{\mathbf{r}},t)$ and $\varphi(\hat{\mathbf{r}},t)$ (electro-magnetic interactions). This concept, applied to different symmetry groups, has been extraordinarily successful in fundamental particle physics, leading us also to the structure of the weak and strong interactions.

10.2 Aharonov Bohm effect

Classically, we clearly would not see any electromagnetic effects if our particle only ever encounters regions with zero electric and magnetic fields. It turns out this is no longer true quantum mechanically, as it is the potentials directly that enter the TDSE.

Let us directly see an example for that:



left: Consider the setup on the left where a particle is trapped on a ring that encircles an infinitely long solenoid magnet, inside of which we have a homogenous magnetic field $\mathbf{B} = B_0 \mathbf{e}_z$. Outside the solenoid $\mathbf{B} = 0$, but there is a non-vanishing vector potential $\mathbf{A} = \frac{\Phi_0}{2\pi r}\mathbf{e}_\varphi$, see yellow box below.

We shall see that this is an example where physically relevant quantities DO depend on the potentials even though the fields are zero wherever the particle is. We assume we only require the azimuthal coordinate φ for the particle (i.e. we fix $r = b$ and $\theta = \pi/2$ in spherical polar coordinates, see diagram above). The TISE using (10.1) with the 3D gradient in the momentum operator then becomes

$$\frac{1}{2m}\left[\left(-i\frac{\hbar}{b\sin(\pi/2)}\frac{\partial}{\partial\varphi}\mathbf{e}_\varphi - q\mathbf{A}(\hat{\mathbf{r}})\right)^2 + q\varphi(\hat{\mathbf{r}})\right]\phi(\varphi) = E\phi(\varphi),$$

$$\frac{1}{2m}\left[-\frac{\hbar^2}{b^2}\frac{d^2}{d\varphi^2} + i\frac{\hbar q\Phi_0}{\pi b^2}\frac{d}{d\varphi} + \left(\frac{q\Phi_0}{2\pi b}\right)^2\right]\phi(\varphi) = E\phi(\varphi) \tag{10.7}$$

We define the shorthand $\beta = \frac{q\Phi_0}{2\pi\hbar}$, then you can convince yourself by back-substitution[25], that Eq. (10.7) is solved by

$$\phi(\varphi) = e^{i\lambda\varphi},$$

$$\lambda = \beta \pm \frac{b}{\hbar}\sqrt{2mE} \stackrel{!}{=} n \in \mathbb{Z},$$

$$E_n = \frac{\hbar^2}{2mb^2}\left(n - \frac{q\Phi_0}{2\pi\hbar}\right)^2. \tag{10.8}$$

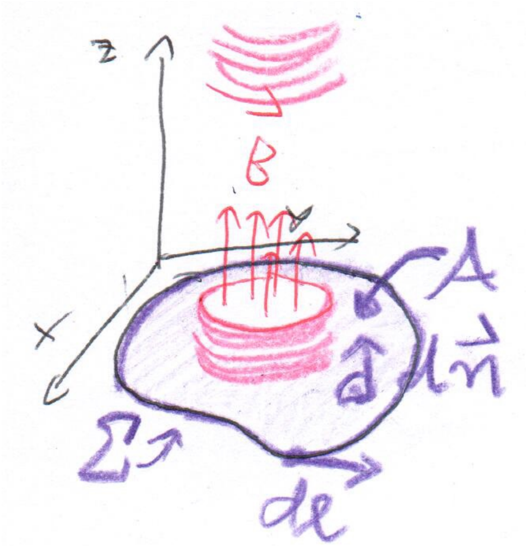
The requirement for λ to be an integer arises so that the wavefunction $\phi(\varphi)$ is single valued/continuous. We could also have solved the "TISE on a ring" (10.7) entirely without any solenoid, in which case we would have found two degenerate solutions $\sim e^{\pm i\lambda\phi}$ for each energy E. Thus inserting the solenoid has lifted the degeneracy of eigenstates and modified the states themselves, even though the magnetic field is zero everywhere where the particles are. See example 76 for the interpretation of this puzzling fact.

[25] or find a few more steps in Griffith.

Vector potential of a solenoid: You learnt in electro-magnetism courses that the magnetic field inside an infinitely long solenoid is $\mathbf{B} = B_0 \mathbf{e}_z$ (with $B_0 = \mu_0 n I$, where n is the number of loops per unit length and I the current, but that is not important here). We also know $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$.
Now we apply Stoke's theorem, that for any vector field $\mathbf{v}$

$$\int_{\mathcal{A}} (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathbf{dn} = \oint_{\Sigma(\mathcal{A})} \mathbf{v} \cdot \mathbf{dl}, \tag{10.9}$$

where $\mathcal{A}$ is a 2D area, Σ the 1D surface encircling it, $\mathbf{dn}$ the infinitesimal area element proportional to a unit vector orthogonal on the surface and $\mathbf{dl}$ an infenitesimal tangent vector on Σ.



left: In our present scenario all those quantities are is shown on the left. We have to choose $\mathcal{A}$ as a circular are centered on the solenoid, so that Σ is also a circle. We then also make an azimuthal Ansatz for the vector potential $\mathbf{A} = A(r)\mathbf{e}_\varphi$.

Choosing $\mathbf{v} = \mathbf{A}$ we have (for $r > d$)

$$\int_{\mathcal{A}} (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{dn} = \int_{\mathcal{A}} \mathbf{B} \cdot \mathbf{dn} = \Phi_0 \stackrel{!}{=} (2\pi r) A(r), \tag{10.10}$$

where $\Phi_0 = B_0(\pi d^2)$ is the magnetic flux through the solenoid, we reach

$$\mathbf{A} = \frac{\Phi_0}{2\pi r} \mathbf{e}_\varphi, \tag{10.11}$$

which importantly <u>does not</u> vanish outside the solenoid, even though the field does.

The discussion above forms the basis of the

Example 76, Aharonov Bohm effect: Instead of finding eigenstates of a particle on the ring, assume a beam of electrons is split into two parts, moved on either side of the solenoid and then recombined. Through a discussion very similar to the above (see Griffith for math steps and a diagram), one sees that the electrons passing the solenoid on the left will arrive with a phase shift of

$$\phi = q\frac{\Phi_0}{\hbar} \tag{10.12}$$

relative to the ones passing on the right, that can be measured by interfering the two beams on a screen. Again this depends on the enclosed flux Φ_0, even though the electrons never enter a region in which there is a magnetic field.
This was experimentally confirmed by Chambers *et. al* [PRL **5** 3 (1960)] using an electron microscope type setup, measuring interference fringes that depend on the enclosed flux despite the electrons never experiencing a magnetic field. We thus have found one more case where our intuition based on classical mechanics leads us astray: Quantum mechanically the vector potential has significance all by itself that it does not have classically!