PHY 303, I-Semester 2023/24, Tutorial 6

18th Oct 2021

This sheet is as usual too large. Please pick whichever topic you agree on practicing first.

Stage 1 (Atoms in a waveguide) At very low temperatures, it is possible to trap atoms in a laser-beam through a technique called optical trapping. Assuming a laser beam that is axially symmetric around the z-axis as shown below, we can approximately write the potential as:

$$V(\mathbf{r}) = \frac{1}{2} m \omega_\perp^2 \left(x^2 + y^2\right). \tag{1}$$

(note that it does not depend on z).

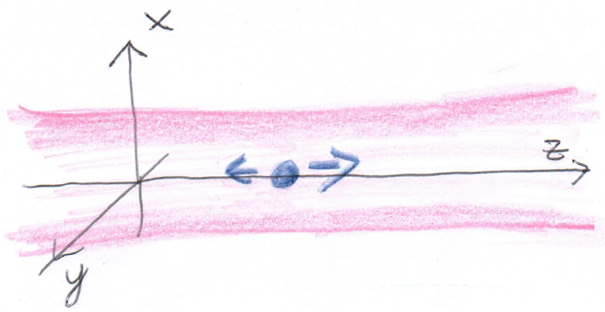

Figure 1: The atom (blue) is attracted to high laser intensity (pink), which drops with $r = \sqrt{x^2 + y^2}$, thus the laser beam along the z axis forms a waveguide for the atom, in which it can move freely along z (arrows).

(a) Adapt the discussion of week8, section 4.1.1. to this scenario: What are the three one-dimensional TISEs which are equivalent to the 3D one? What form do the eigenstates take? Which quantum numbers control them and what do they physically imply. Give an equation for the energy.

(b) Now consider only states for which the energy difference ΔE to the groundstate is $\Delta E \ll \hbar\omega_\perp$, which subset of states from (a) does this condition select?

(c) Suppose you want to describe an atom in this waveguide, localized near a position $z = z_0$ in the waveguide and moving with a velocity v_z in the z-direction. Which 3D wavefunction would describe such an atom?

(d) Rewrite the 3D expectation value of the operator for the gravitational potential energy V_{grav} in a state such as what you found in (c). Lets consider two cases

$$\text{(i)} \qquad V_{\text{grav}}(\mathbf{r}) = mgz \qquad \text{(direction of gravity along z)}, \tag{2}$$

$$\text{(ii)} \qquad V_{\text{grav}}(\mathbf{r}) = mgx \qquad \text{(direction of gravity along x)}. \tag{3}$$

You do not need to do any non-trivial 1D integrations. *Hint: Trivial means the answer is 0 or 1.*

Stage 2 (Angular momentum)

(i) How can you prove in quantum mechanics that angular momentum is conserved if the potential is spherically symmetric?

(ii) Is angular momentum also conserved if the potential is not spherically symmetric? Why/why not?

(iii) Show the commutator

$$\left[\hat{L}_x, \hat{L}_y,\right] = i\hbar\hat{L}_z. \tag{4}$$

Do this once based on your knowledge of the commutators of $\hat{r}_k$ and $\hat{p}_\ell$, and once from the definitions via partial derivatives applied onto a testfunction.

Stage 3 (Pictures of time dependence)

(i) Convince yourself that for time-independent Hamiltonian, $|\,\Psi(t)\,\rangle = \hat{U}(t)|\,\Psi(0)\,\rangle$ solves the TDSE. Here $\hat{U}(t) = \exp\left[-i\hat{H}t/\hbar\right]$ is the time-evolution operator or propagator. Discuss what is meant by exponential of an operator and how we could possible find this in practice.

(ii) Write the time-dependence of a generic expectation value $\langle\,\Psi(t)\,|\hat{O}|\,\Psi(t)\,\rangle$ and thus convince yourself that instead of assuming a time-dependent states and time-independent operator, we could also work with a time-independent state and time-dependent Operator. Which is that operator?

Stage 4 (Three-dimensional wavefunctions) Consider a radially symmetric problem, such that an eigenstate of the Hamiltonian takes the form $\phi(\mathbf{r}) = R(r)Y(\theta, \varphi)$.

(i) Why do we know that the eigenstates takes this form?

(ii) Suppose the particle carries a charge q. Then the operator for its electric dipole is

$$\hat{\mathbf{d}} = q\hat{\mathbf{r}}. \tag{5}$$

How can you find the expectation value of this dipole? What integration(s) do you have to do? *Hint: You can actually find the answer without any nasty integrations, but the idea here is to write all the steps converting the 3D integration into separate 1D ones, and only then "see" the answer.*