PHY 303, I-Semester 2023/24, Tutorial 5

4th Oct 2023

Discuss on your table in AIR. When all teams finished a stage, make sure all students at your table understand the solution and agree on one by using the board.

Stage 1 (Postulates) Consider an arbitrary time-independent Hamiltonian, with eigenvalue problem

$$\hat{H}|\,\phi_n\,\rangle = E_n|\,\phi_n\,\rangle, \tag{1}$$

where there is no degeneracy (thus $E_n \neq E_m$ for $n \neq m$). The system is in the quantum state

$$|\,\Psi\,\rangle = \frac{1}{2}|\,\phi_1\,\rangle + \frac{1}{\sqrt{2}}|\,\phi_2\,\rangle + \frac{1}{2}|\,\phi_3\,\rangle \tag{2}$$

and assume $E_1 = 1$ eV, $E_2 = 2$ eV, $E_3 = 8$ eV.

(a) Check that the state is correctly normalized.

(b) We now repeatedly create the state (2) and then measure the energy of the system. Find the expectation value of this measurement, all possible results of of a single measurement and the most likely result of a measurement. With which probability will a single measurement result be the same as the expectation value?

(c) Suppose in a first measurement we have found energy E_2. Now we do a second measurement on the same system without re-initialising the quantum state. Answer all the questions in (b) again.

(d) Now consider the Hermitian operator

$$\hat{O} = \sum_{n=0}^{\infty} o_n \left(|\,\phi_{n+1}\,\rangle\langle\,\phi_n\,| + |\,\phi_n\,\rangle\langle\,\phi_{n+1}\,|\right). \tag{3}$$

with some constants o_n. What is the expectation value of measurements of this operator in the state (2)?

Stage 2 (Commutators) Show the relation (3.41): $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$. Then evaluate the following commutators:

(i) $\left[(m\omega)^2\hat{x}^2 + \hat{p}^2, \hat{x}\right]$

(ii) Consider operators $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$. Assume a Hilbertspace with only two basis states $|\,1\,\rangle$ and $|\,2\,\rangle$. In this basis, let the matrix representations of those operators be

$$\underline{\underline{\sigma_x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \underline{\underline{\sigma_y}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \underline{\underline{\sigma_z}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

Find the matrix representation of all commutators $\left[\hat{\sigma}_k, \hat{\sigma}_n\right]$, and try to find a single neat expression for them.

Stage 3 (Uncertainty relations)

(i) Consider the anharmonic oscillator Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \chi \hat{x}^4. \tag{5}$$

Can one simultaneously know the momentum and energy? If not, what is the uncertainty relation between these two observables?

(ii) For which of the following tasks can you use the energy-time uncertainty relation (and why or why not)? What does it tell you if yes?

(a) A harmonic oscillator is in eigenstate $\phi_5(x)$. We want to know the period of oscillations of the complex phase of the wavefunction.

(b) A harmonic oscillator is in a superposition of eigenstates $\phi_5(x)$, $\phi_6(x)$ and $\phi_7(x)$. We want to approximate the time-scale of oscillations of the expectation value of momentum $\langle \hat{p} \rangle$.

(c) A quantum superposition state at time $t = 0$

$$|\Psi(0)\rangle = \sum_n c_n |\phi_n\rangle \tag{6}$$

contains a superposition of a larger number of basis states as shown in the figure below (left). Because of all the different phase factors $e^{-iE_n t/\hbar}$ in the corresponding time evolving state, the expectation value of some (unspecified) operator $\langle \hat{O} \rangle$ quickly decays to near zero as also shown in the figure (right). However let this be a case where all the energies E_n are rational multiples of each other, then there is a time t_{rev} called "revival time", where $\langle \hat{O} \rangle$ returns to its initial value because all $e^{-iE_n t_{\text{rev}}/\hbar} = 1$.
You want to know the timescale of this revival and the timescale it takes for $\langle \hat{O} \rangle$ to decay to near zero (answer for both of these separately).