PHY 303, I-Semester 2023/24, Tutorial 4 solution

27. Sept. 2023

Work in the same teams as for assignments. Do "Stages" in the order below. Discuss via online (video or audio) conference on a subchannel for your group.

Stage 1 Catch up on all items of tutorial 3 that interest you and you did not yet have time to go through.

Stage 2 Discuss the following clarifying exercises depending on which topic you feel you require more clarification:

(i) (ladder operators) Explicitly with all details, the first three Harmonic oscillator eigenstates are

$$\phi_0(x) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{-x^2/2\sigma^2} \tag{1}$$

$$\phi_1(x) = \frac{\sqrt{2}}{(\pi\sigma^2)^{1/4}} \left(\frac{x}{\sigma}\right) e^{-x^2/2\sigma^2} \tag{2}$$

$$\phi_2(x) = \frac{1}{\sqrt{2}} \frac{1}{(\pi\sigma^2)^{1/4}} \left(\frac{2x^2}{\sigma^2} - 1\right) e^{-x^2/2\sigma^2} \tag{3}$$

Using the ladder operators

$$\hat{\hat{a}}_\pm = \frac{1}{\sqrt{2\sigma^2}} \left(x \mp \sigma^2 \frac{\partial}{\partial x}\right), \tag{4}$$

(these slightly differ from those in the lecture/Griffith, but fulfill the same job), show that:

$$\begin{aligned} \hat{\hat{a}}_+ \phi_n(x) &= \sqrt{n+1}\phi_{n+1}(x), \\ \hat{\hat{a}}_- \phi_n(x) &= \sqrt{n}\phi_{n-1}(x), \end{aligned} \tag{5}$$

explicitly among those first three states.

solution:

$$\phi_0(x) = \left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{-x^2/2\sigma^2}$$

$$\phi_1(x) = \frac{\sqrt{2}}{(\pi\sigma^2)^{1/4}} \left(\frac{x}{\sigma}\right) e^{-x^2/2\sigma^2}$$

$$\phi_2(x) = \frac{1}{\sqrt{2}} \frac{1}{(\pi\sigma^2)^{1/4}} \left(\frac{2x^2}{\sigma^2} - 1\right) e^{-x^2/2\sigma^2}$$

$$a_+\phi_0(x) = \frac{1}{\sqrt{2\sigma^2}} \left(x - \sigma^2 \frac{\partial}{\partial x}\right) \left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{-x^2/2\sigma^2}$$

$$= \left(\frac{1}{\sqrt{2\sigma^2}}\right) \left(\frac{1}{\pi\sigma^2}\right)^{1/4} \left[x - \sigma^2 \left(\frac{-2x}{2\sigma^2}\right)\right] e^{-x^2/2\sigma^2} = \frac{\sqrt{2}}{(\pi\sigma^2)^{1/4}} \left(\frac{x}{\sigma}\right) e^{-x^2/2\sigma^2} = \phi_1(x)$$

$$a_-\phi_0(x) = \frac{1}{\sqrt{2\sigma^2}}\left(x+\sigma^2\frac{\partial}{\partial x}\right)\left(\frac{1}{\pi\sigma^2}\right)^{1/4} e^{-x^2/2\sigma^2}$$
$$= \left(\frac{1}{\sqrt{2\sigma^2}}\right)\left(\frac{1}{\pi\sigma^2}\right)^{1/4}\left[x+\sigma^2\left(\frac{-2x}{2\sigma^2}\right)\right] e^{-x^2/2\sigma^2} = 0$$

$$\begin{aligned}
a_+\phi_1(x) &= \frac{1}{\sqrt{2\sigma^2}}\left(x-\sigma^2\frac{\partial}{\partial x}\right)\frac{\sqrt{2}}{(\pi\sigma^2)^{1/4}}\left(\frac{x}{\sigma}\right) e^{-x^2/2\sigma^2}\\
&= \frac{1}{\sigma^2}\frac{1}{(\pi\sigma^2)^{1/4}}\left(x^2-\sigma^2+x^2\right) e^{-x^2/2\sigma^2}\\
&= \frac{1}{(\pi\sigma^2)^{1/4}}\left(\frac{2x^2}{\sigma^2}-1\right) e^{-x^2/2\sigma^2} = \sqrt{2}\phi_2(x)
\end{aligned}$$

$$\begin{aligned}
a_-\phi_1(x) &= \frac{1}{\sqrt{2\sigma^2}}\left(x+\sigma^2\frac{\partial}{\partial x}\right)\frac{\sqrt{2}}{(\pi\sigma^2)^{1/4}}\left(\frac{x}{\sigma}\right) e^{-x^2/2\sigma^2}\\
&= \frac{1}{\sigma^2}\frac{1}{(\pi\sigma^2)^{1/4}}\left(x^2+\sigma^2-x^2\right) e^{-x^2/2\sigma^2} = \frac{1}{(\pi\sigma^2)^{1/4}} e^{-x^2/2\sigma^2} = \phi_0(x)
\end{aligned}$$

$$\begin{aligned}
a_+\phi_2(x) =& \frac{1}{\sqrt{2\sigma^2}}\left(x-\sigma^2\frac{\partial}{\partial x}\right)\frac{1}{\sqrt{2}}\frac{1}{(\pi\sigma^2)^{1/4}}\left(\frac{2x^2}{\sigma^2}-1\right) e^{-x^2/2\sigma^2}\\
=& \frac{1}{2\sigma(\pi\sigma^2)^{1/4}}\left(x-\sigma^2\frac{\partial}{\partial x}\right)\left(\frac{2x^2}{\sigma^2}-1\right) e^{-x^2/2\sigma^2}\\
=& \frac{1}{2\sigma(\pi\sigma^2)^{1/4}}\left(\frac{2x^3}{\sigma^2}-x-4x+\frac{2x^3}{\sigma^2}-x\right) e^{-x^2/2\sigma^2}\\
=& \frac{1}{2\sigma(\pi\sigma^2)^{1/4}}\left(\frac{4x^3}{\sigma^2}-6x\right) e^{-x^2/2\sigma^2} = \sqrt{3}\phi_3(x)
\end{aligned}$$

$$\begin{aligned}
a_-\phi_2(x) =& \frac{1}{\sqrt{2\sigma^2}}\left(x+\sigma^2\frac{\partial}{\partial x}\right)\frac{1}{\sqrt{2}}\frac{1}{(\pi\sigma^2)^{1/4}}\left(\frac{2x^2}{\sigma^2}-1\right) e^{-x^2/2\sigma^2}\\
=& \frac{1}{2\sigma(\pi\sigma^2)^{1/4}}\left(x+\sigma^2\frac{\partial}{\partial x}\right)\left(\frac{2x^2}{\sigma^2}-1\right) e^{-x^2/2\sigma^2}\\
=& \frac{1}{2\sigma(\pi\sigma^2)^{1/4}}\left(\frac{2x^3}{\sigma^2}-x+4x-\frac{2x^3}{\sigma^2}+x\right) e^{-x^2/2\sigma^2}\\
=& \frac{2x}{\sigma(\pi\sigma^2)^{1/4}} e^{-x^2/2\sigma^2} = \sqrt{2}\phi_1(x)
\end{aligned}$$

(ii) (wavepacket spreading) Consider a free particle treated quantum mechanically, that is at $t=0$ in a Gaussian wavefunction, Eq. (2.82), at $x_0=0$ with spatial width σ. For the following examples, find the time $t_{\rm spr}>0$ at which the spatial uncertainty has increased by a factor of two.

solution: *using equation (2.90):*

$$\sigma_x(t) = \sqrt{\sigma^2 + \frac{\hbar^2 t^2}{m^2\sigma^2}}$$
$$\sigma_x(t) = 2\sigma(Given)$$
$$\Rightarrow \quad 2\sigma = \sqrt{\sigma^2 + \frac{\hbar^2 t^2}{m^2\sigma^2}} \Rightarrow t = \frac{\sqrt{3}m\sigma^2}{\hbar}$$

(a) An electron with $m_e = 9.1 \times 10^{-31}$ kg and an uncertainty of the size of an atom $\sigma = 5.3 \times 10^{-11}$ m.
using the result for electron: $t_{spr} = \frac{\sqrt{3}m_e\sigma^2}{\hbar} = \frac{\sqrt{3}(9.1\times10^{-31})(5.3\times10^{-11})^2}{1.05\times10^{-34}} = 4.188 \times 10^{-17}s$

(b) A cricket ball of mass $m = 0.15$ kg, with $\sigma = 1$ μm.
$t_{spr} = 2.4 \times 10^{21}s = 8 \times 10^{13}$ *years. [compare the age of the universe* $\tau = 14 \times 10^9$ *years.*

(c) An electron with $m_e = 9.1 \times 10^{-31}$ kg and an uncertainty of the size of a nanoparticle $\sigma = 20$ nm.
$t_{spr} = 6.004 \times 10^{-12}s$

Stage 3 Bra-Ket Notation: Make the usage instructions for Dirac (Bra-Ket) notation from week 6 accessible on a device.

(a) Then translate the following superposition state wavefunction (assuming the standard particle in the infinite square well potential) into Dirac notation:

$$\Psi(x) = \frac{1}{2}\phi_0(x) + \frac{1}{\sqrt{2}}\phi_1(x) + \frac{1}{2}\phi_2(x). \tag{6}$$

(b) Only using Dirac notation, check whether this state is normalised correctly.

(c) Using Dirac notation, find the probabilities for energy to be E_n, for position to be within $[x, x + dx]$ and for momentum to be within $[p + dp]$. For which cases and how/when do we have to convert back to a more explicit representation of the quantum states at some point, to get an answer? Do that conversion without the final step of evaluating any integrals that pop up.

(d) Write equation (1.71) and other equations required to use it in Dirac notation.

solution:

(a) In Dirac notation we can represent the eigenstate as $|\,\phi_n\,\rangle$*, so the wavefuction* $\Psi(x)$ *will now be represented by a vector*

$$|\,\Psi\,\rangle = \frac{1}{2}|\,\phi_0\,\rangle + \frac{1}{\sqrt{2}}|\,\phi_1\,\rangle + \frac{1}{2}|\,\phi_2\,\rangle. \tag{7}$$

(b) *To check for normalization we have*

$$\begin{aligned}\langle \Psi \mid \Psi \rangle &= \left[\frac{1}{2}\langle \phi_0 \mid + \frac{1}{\sqrt{2}}\langle \phi_1 \mid + \frac{1}{2}\langle \phi_2 \mid\right]\left[\frac{1}{2}\mid \phi_0 \rangle + \frac{1}{\sqrt{2}}\mid \phi_1 \rangle + \frac{1}{2}\mid \phi_2 \rangle\right] \\ &= \frac{1}{4}\langle \psi_0 \mid \psi_0 \rangle + \frac{1}{2}\langle \psi_1 \mid \psi_1 \rangle + \frac{1}{4}\langle \psi_2 \mid \psi_2 \rangle \\ &= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1.\end{aligned}$$

where in the second equality we have used that eigenstates with different index are orthogonal and hence e.g. $\langle \phi_0 \mid \phi_1 \rangle = 0$.

(c) *To find out the probability for energy to be* E_n*, we need to project the state vector (*$\mid \Psi \rangle$*) to our energy eigenvectors (*$\mid \phi_n \rangle$*), and its square will give the probability. So we have*

$$\begin{aligned}p_n &= |c_n|^2 \\ &= |\langle \phi_n \mid \Psi \rangle|^2,\end{aligned}$$

and find energies $[E_0, E_1, E_2]$ *with probabilities* $[\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$ *respectively.*

Similarly to find the probability for position to be within $[x, x + dx]$*. We project the state vector to position basis (*$\mid x \rangle$*)*

$$\begin{aligned}p(x) &= |\langle x \mid \Psi \rangle|^2\, dx = |\Psi(x)|^2\, dx \\ &= \left(\frac{1}{2}\phi_0(x) + \frac{1}{\sqrt{2}}\phi_1(x) + \frac{1}{2}\phi_2(x)\right)^* \left(\frac{1}{2}\phi_0(x) + \frac{1}{\sqrt{2}}\phi_1(x) + \frac{1}{2}\phi_2(x)\right) dx \\ &= \left[\frac{1}{4}|\phi_0(x)|^2 + \frac{1}{2}|\phi_1(x)|^2 + \frac{1}{4}|\phi_2(x)|^2\right. \\ &\quad \left. + \left(\frac{1}{2\sqrt{2}}\phi_0(x)^*\phi_1(x) + \frac{1}{4}\phi_0(x)^*\phi_2(x) + \frac{1}{\sqrt{2}}\phi_1(x)^*\phi_2(x) + c.c.\right)\right] dx.\end{aligned}$$

This cannot be further simplified, and its evaluation would require the explicit form of $\phi_n(x)$ *known from the lecture. Note that the last line is nonzero since there is no integration (only* $\int \phi_1^*(x)\phi_2(x) = 0$ *NOT* $\phi_1^*(x)\phi_2(x)$ *itself.*

Lastly for the momentum probability within $[p, p + dp]$ *we project to momentum eigenvectors (*$\mid p \rangle$*)*

$$p(p) = |\langle p \mid \Psi \rangle|^2$$

in a similar way.

(d) *The eqn* (1.71) *in terms of Dirac notation can be written as*

$$\mid \Psi(t) \rangle = \sum_n c_n(t) \mid \phi_n \rangle = \sum_n c_n(0) e^{-iE_n t/\hbar} \mid \phi_n \rangle \qquad (8)$$

Where $c_n(0)$ *in Dirac notation is given by*

$$c_n(0) = \langle \phi_n \mid \Psi \rangle$$