PHY 303, I-Semester 2023/24, Assignment 6

Instructor: Sebastian Wüster
Due-date: 12 Nov 2023

(1) Angular momentum and commutators [10pts]:

(a) Consider the three dimensional Gaussian wavepacket given in Eq. (4.15) of the lecture, with $\mathbf{r}_0 = [x_0, 0, 0]^T$ and $\mathbf{k}_0 = [0, k_0, 0]^T$. Make a sketch of the probability density for $\sigma \ll x_0$ and describe the state of the particle in terms of physics. Then find the expectation value of the angular momentum and discuss. *Hint: Make extensive use of symmetries to avoid most of the integrations that pop up.* Why does this result make sense?

(b) Consider an eigenstate of the TISE with a spherically symmetric potential. Show explicitly (with integrations) that in any eigenstate $\langle \hat{\mathbf{r}} \rangle = 0$ and $\langle \hat{\mathbf{p}} \rangle = 0$ and then argue why we could have known this without calculation.

(c) Show the commutators [3 pts]

$$\left[\hat{\mathbf{L}}^2, \hat{L}_n\right] = 0, \tag{1}$$
$$\left[\hat{L}_z, \hat{L}_\pm\right] = \pm\hbar\hat{L}_\pm, \tag{2}$$
$$\left[\hat{\mathbf{L}}^2, \hat{L}_\pm\right] = 0. \tag{3}$$

and show that

$$\hat{\mathbf{L}}^2 = \hat{L}_\pm\hat{L}_\mp + \hat{L}_z^2 \mp \hbar\hat{L}_z. \tag{4}$$

(d) Suppose that $\left[\hat{A}, \hat{B}\right] = c \in \mathbb{C}$ is just a number. Consider a function $f(x)$ with convergent Taylor series and its derivative $f'(x) = df(x)/dx$. Show that $\left[f(\hat{A}), \hat{B}\right] = cf'(\hat{A})$.

(2) Two-dimensional Harmonic oscillator in polar coordinates [10pts]: Consider the isotropic quantum harmonic oscillator in two dimensions using planar polar coordinates.

(a) Convert the Hamiltonian from cartesian coordinates:

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2), \tag{5}$$

to planar polar coordinates $r = \sqrt{x^2 + y^2}$, $\varphi = \text{atan2}(y, x)$.

(b) With the Ansatz $\phi_{n\ell}(r, \varphi) = \Phi_{n\ell}(r)A_\ell(\varphi)$, use separation of variables to split the TISE into an angular and a radial part. Solve the angular equation, using techniques and boundary conditions similar to what you learnt regarding angular momentum states in 3D. Discuss how one would go about solving the radial equation schematically (you do not have to solve it).

(3) Two-dimensional Harmonic oscillator in the Schrödinger and Heisenberg pictures [10pts]:

(a) Write down all eigenstates and eigenenergies of the Hamiltonian (5) by adapting the discussion of section 4.1.1.

(b) Consider the initial state

$$\Psi(x, y, t = 0) = \frac{1}{2}\big[\phi_0(x) + \phi_1(x)\big]\big[\phi_0(y) + i\phi_1(y)\big]. \tag{6}$$

Visualise this state (e.g. with `mathematica`) and discuss its physical meaning. Then find the time evolution of $\langle \hat{\mathbf{r}} \rangle$ in the Schrödinger picture and interpret/discuss the result.

(c) Now, let us reproduce these results in the Heisenberg picture. Start by finding the Heisenberg equations of motion for operators $\hat{x}_H(t)$, $\hat{y}_H(t)$, $\hat{p}_{xH}(t)$, $\hat{p}_{yH}(t)$.

(d) Solve those, to find a solution for the time-dependent operators in terms of operators at time $t = 0$. *Hint: To solve differential equations involving operators, pretend they are not operators initially, then confirm the solution holds also if they are, possibly worrying about commutators.*

(e) Recalculate $\langle \hat{\mathbf{r}} \rangle$ in the Heisenberg picture and confirm your result from (c).

(4) Circular Hydrogen states: [**10pts**] The code `Assignment6_program_draft_v1.nb` can setup all electronic eigenfunctions of the Hydrogen atom.

(a) Visualise the probability density, and the complex phase in what is called a "circular Rydberg state" $n = 10$, $l = 9$, $m = 9$ using smartly chosen 2D or 1D cuts through those.

(b) Calculate the probability current density in that state. With all information together, discuss the corresponding physical state of the electron.

(c) Discuss the corresponding charge and current density in this state (without calculation), their time-dependences, and finally how you can reconcile all earlier results in this question with a stable atom.