PHY 303, I-Semester 2023/24, Assignment 2

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Due-date: 27. Aug 2023

(1) Functions spaces and operators:

(a) Show that the function space $\mathbb{L}_2$ over the field of complex numbers as defined in section 1.5.4. is a vectorspace, by addressing all the points in the definition of section 1.5.3. Comment carefully on all the points, and in particular on whether results of operations are in $\mathbb{L}_2$ again. [1pt]

(b) Show that $\langle(\hat{O} - \langle\hat{O}\rangle)^2\rangle = \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$. [1pt]

(c) Show that the momentum operator is Hermitian, using Eq. (1.24). [2pts]

(d) Show that, if you have an orthonormal basis of a function vector space, using Eq. (1.18), you can find the coefficients as $f_n = (b_n, f)$, using the scalar product in (1.17). [1pt]

(e) Consider the step-wise function $f(x) = h = const$ for $-\frac{S}{4} \leq x \leq \frac{S}{4}$, $f(x) = 0$ for other points in the interval $-S \leq x \leq S$ and then infinitely repeated outside this interval, as drawn in Fig. 1. What is the period of this function?.

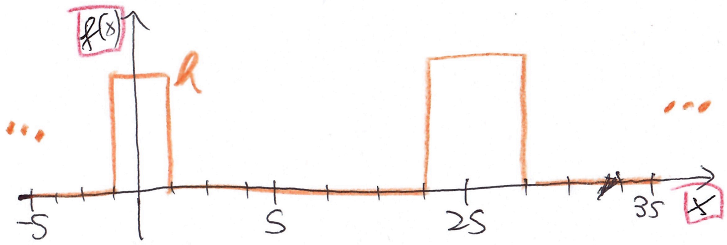

Figure 1: Stepwise function $f(x)$ (orange) as defined in the text.

You can show that $b_n = \cos\left(\frac{2\pi n}{L}x\right)$ form a basis for the vector-space of all symmetric functions with period L. Normalise this basis on its interval of periodicity, then use this and your result of part (1d) above to explicitly find the coefficients b_n in the basis expansion of the function $f(x)$, i.e. writing

$$f(x) = \sum_{n=0}^{\infty} f_n \cos\left(\frac{2\pi n}{L}x\right). \tag{1}$$

[5pts] *Hint, for periodic functions, we use a scalar product in which we integrate over one period only. You may check results of integrations with a computer, but I expect you to also learn the manual method as well. You can also check your coefficients using the script provided in assignment 1.*

(2) Time-dependence of a quantum harmonic oscillator:

(a) Write down the TDSE for a particle of mass m in a harmonic potential, such that the particle would classically oscillate with frequency ω [1 pt]

(b) Show that

$$\Psi(x,t) = \frac{1}{(\sigma^2\pi)^{1/4}} e^{-i\frac{\omega t}{2}} e^{-\frac{[x-x_0\cos(\omega t)]^2}{2\sigma^2}} e^{-i\left(\frac{x_0}{\sigma}\sin(\omega t)\right)\left(\frac{x-x_0\cos(\omega t)/2}{\sigma}\right)} \tag{2}$$

solves that TDSE. [5 pts]

(c) Find the expectation value of the position, expectation value of the momentum, and discuss their time evolution in terms of Ehrenfest's theorem [2 pts].

(d) Finally find the probability density for this particle as a function of time, and discuss its physical meaning [2 pts].

(3) Heisenberg's uncertainty principle: [10pts] Show that the wavefunction Eq. (2) of question 2 satisfies Heisenbergs uncertainty principle at all times. How do uncertainties change with time?

(4) Particle bouncing in an infinite square well: Consider a particle of mass m in the infinite square well potential discussed in section 2.2.1. of the lecture. At time $t = 0$, let its wavefunction be

$$\Psi(x, t = 0) = \sqrt{\frac{1}{a}}\left(\sin(\pi x/a) - \sin(2\pi x/a)\right). \tag{3}$$

(4a) Find the wavefunction at any later time $t > 0$. [2 pts]

(4b) From that calculate the probabilities that the particle is in the left side of the well (at $0 < x < a/2$) or the right side ($a/2 < x < a$) [3 pts]

(4c) Calculate the probability current at $x = a/2$ as a function of time, and relate your finding with the answer to (4b). [3 pts]

(4d) What is the probability to measure the particle to have energy $E = \pi^2\hbar^2/(2ma^2)$ at $t = 0$? What about later times $t > 0$? *Hint: I have not actually told you the rule yet, let's make it part of the assignment to FIND the rule: Suppose you have found all solutions of the TISE $\hat{H}\phi_n(x) = E_n\phi_n(x)$. Now you can write ANY wavefunction as $\Psi(x) = \sum_n c_n\phi_n(x)$. Find the energy expectation value and based on that argue what logically should be the probability for the system to have energy E_n ?* [1 pt]

(4e) Suppose we have measured the energy to be $E = \pi^2\hbar^2/(2ma^2)$ at time $t_m = (2ma^2)/(\pi\hbar)$ and subsequently measure it again at $t = 2t_m$. What is the probability to find the same value E again? [1 pt] *Hint: Think about what we told you about position measurements, and then generalise this to energy measurements.*