PHY 303, I-Semester 2021/22, Assignment 2

Instructor: Sebastian Wüster
Due-date: 28. Aug 2021

(1) Functions spaces and operators:

(a) Show that the function space $\mathbb{L}_2$ over the field of complex numbers as defined in 1.5.4. is a vectorspace, following all items in the definition in section 1.5.3. Comment carefully on all, and in particular on whether results of operations are in $\mathbb{L}_2$ again. [1pt]

Solution: Let $f(x), g(x), h(x)$ belong to $\mathbb{L}_2$ the set of all square integrable function such that they follow:

$$\int dx|f(x)|^2 < \infty$$

For $\mathbb{L}_2$ to be a vector space:
1) *Associative property: $u + (v + w) = (u + v) + w$. Clearly we have*

$$[f(x) + h(x)] + g(x) = f(x) + [h(x) + g(x)]$$

Also, if $\int dx\,|f(x)|^2 = C < \infty$ and $\int dx\,|g(x)|^2 = D < \infty$, then also

$$\int dx|f(x) + g(x)|^2 \overset{\textit{triangle inequality}}{\leq} \int dx(|f(x)|^2 + |g(x)|^2)$$

$$= \int dx\,|f(x)|^2 + \int dx\,|g(x)|^2 = C + D < \infty,$$

so the result of any "+" operations between functions in $\mathbb{L}_2$ is again in $\mathbb{L}_2$.
2) *Commutativity: $u + v = u + v$*

$$f(x) + g(x) = g(x) + f(x)$$

3) *Identity element: We require a*

$$f(x) + h(x) = f(x).$$

That works out if we chose $h(x) \equiv 0$, i.e. a constant function that is always zero.
4) *Inverse element: There needs to be an element $g(x)$ such that*

$$f(x) + g(x) = 0$$

if we choose $g(x) = -f(x)$.
5) *Distributivity of scalar multiplication*
Let a, b be scalars, then clearly

$$a[f(x) + g(x)] = af(x) + ag(x)$$

and

$$(a+b)f(x) = af(x) + bf(x).$$

We can also again check that $\int dx\,|af(x)|^2 = |a|^2 C < \infty$, *so also scalar multiplication results in an element that remains in* $\mathbb{L}_2$.

6) *Multiplicative identity: The multiplicative identity in the field is just the number "1". Clearly*

$$1 \times f(x) = f(x).$$

7) *Scalar multiplication with field multiplication*
Let a,b be scalars, then

$$a(bf(x)) = (ab)f(x)$$

Thus, $\mathbb{L}_2$ *is a vectorspace.*

(b) Show that $\langle(\hat{O} - \langle\hat{O}\rangle)^2\rangle = \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$. [1pt]
Solution:

$$\begin{aligned}\langle(\hat{O} - \langle\hat{O}\rangle)^2\rangle &= \langle(\hat{O}^2) + \langle\hat{O}\rangle^2 - 2\hat{O}\langle\hat{O}\rangle\rangle \\ &= \langle\hat{O}^2\rangle + \langle\hat{O}\rangle^2 - 2\langle\hat{O}\rangle\langle\hat{O}\rangle \\ &= \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2\end{aligned}$$

(c) Show that the momentum operator is Hermitian, using Eq. (1.24). [2pts]
Solution: The operator $\hat{A}$ *is hermitian if:*

$$\int_{-\infty}^{\infty} (\hat{A}\Psi)^* \Psi dx = \int \Psi^* \hat{A}\Psi dx$$

For $\hat{p}$ *to be hermitian :*

$$\int_{-\infty}^{\infty} (\hat{p}\Psi)^* \Psi dx = \int_{-\infty}^{\infty} \Psi^* \hat{p}\Psi dx$$

$$where,\ \hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\begin{aligned}\int_{-\infty}^{\infty} (\hat{p}\Psi)^* \Psi dx &= \int_{-\infty}^{\infty} (-i\hbar\frac{d\Psi}{dx})^* \Psi dx \\ &= i\hbar\int_{-\infty}^{\infty} (\frac{d\Psi}{dx})^* \Psi dx\end{aligned}$$

Using Integrating by parts

$$= i\hbar\left([\Psi\Psi^*]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Psi^* \frac{d\Psi}{dx} dx\right)$$

As $x \to \pm\infty, \psi(x) \to 0$, *hence*

$$= -\int_{-\infty}^{\infty} \Psi^* \left(i\hbar \frac{d\Psi}{dx} \right) dx$$
$$= \int_{-\infty}^{\infty} \Psi^* \hat{p} \Psi dx$$

So, $\hat{p}$ *is Hermitian.*

(d) Show that, if you have an orthonormal basis of a function vector space, using Eq. (1.18), you can find the coefficients as $f_n = (b_n, f)$, using the scalar product in (1.17). [1pt]
Solution:

$$f(x) = \sum_{n=0}^{\infty} f_k b_k(x) \tag{1}$$

$$(g, f) = \int_{-\infty}^{\infty} dx\, g^*(x) f(x) \tag{2}$$

Using Eq.1 and Eq.2

$$(b_n, f) = \int_{-\infty}^{\infty} dx b_n^*(x) f(x) = \int_{-\infty}^{\infty} dx b_n^*(x) \sum_{k=0}^{\infty} f_k b_k(x) = \sum_{k=0}^{\infty} f_k \underbrace{\int_{-\infty}^{\infty} dx b_n^*(x) b_k(x)}_{=\delta_{nk}} = f_n \tag{3}$$

(e) Consider the step-wise function $f(x) = h = const$ for $-\frac{S}{4} \leq x \leq \frac{S}{4}$, $f(x) = 0$ for other points in the interval $-S \leq x \leq S$ and then infinitely repeated outside this interval, as drawn in Fig. 1. What is the period of this function?.

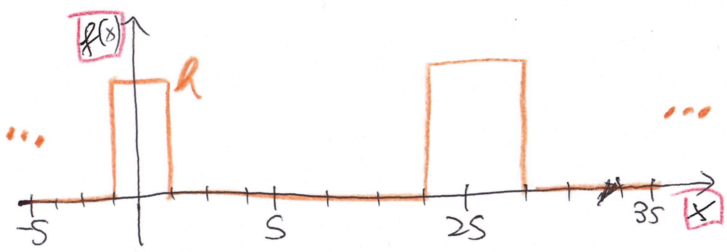


Figure 1: Stepwise function $f(x)$ (orange) as defined in the text.

You can show that $b_n = \cos\left(\frac{2\pi n}{L} x\right)$ form a basis for the vector-space of all symmetric functions with period L. Normalise this basis on its interval of periodicity, then use this and your result of part (1d) above to explicitly find the coefficients b_n in the basis expansion of the function $f(x)$, i.e. writing

$$f(x) = \sum_{n=0}^{\infty} f_n \cos\left(\frac{2\pi n}{L}x\right). \tag{4}$$

[5pts] *Hint, for periodic functions, we use a scalar product in which we integrate over one period only. You may check results of integrations with a computer, but I expect you to also learn the manual method as well. You can also check your coefficients using the script provided in assignment 1.*

Solution: The function in Fig. 1 defined in the question has a period 2S, since $f(x+2S) = f(x)$ for all x. First, let us normalize the basis $\bar{b}_n(x) = \cos\left(\frac{\pi n}{S}x\right)$ across this period;

For n = 0, the normalisation integral for the first cosine is $\int_{-S}^{S} dx \cos(0) = 2S$, hence we write a normalized basis function $\bar{b}_0(x) = \frac{1}{\sqrt{2S}}$. For higher n we need to integrate e.g.

$$\int_{-S}^{S} dx\, |b_n(x)|^2 = \int_{-S}^{S} dx\, \underbrace{\cos(\pi nx/S)}_{\equiv g(x)} \underbrace{\cos(\pi nx/S)}_{\equiv h'(x)}$$

$$\stackrel{I.b.P.}{=} \underbrace{\underbrace{\cos(\pi nx/S)\left(\frac{S}{\pi n}\right)\sin(\pi nx/S)}_{g(x)h(x)}\Bigg|_{-S}^{S}}_{=0} - \int_{-S}^{S} dx\, \underbrace{\underbrace{\left[-\sin^2(\pi nx/S)\right]}_{g'(x)h(x)}}_{=-[1-\cos^2]}.$$

$$\Leftrightarrow 2\int_{-S}^{S} dx\, \cos^2(\pi nx/S) = \int_{-S}^{S} dx\, 1 \Leftrightarrow \int_{-S}^{S} dx\, \cos^2(\pi nx/S) = S.$$

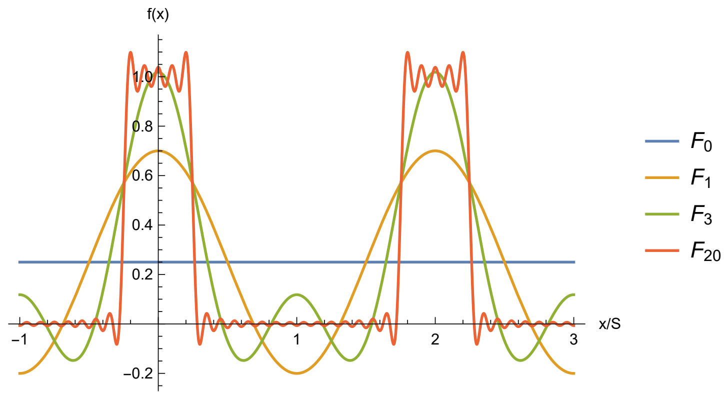
thus all the $n > 0$ basis functions are normalized as $\bar{b}_n(x) = \sqrt{\frac{1}{S}}\cos(\pi nx/S)$.

To finally expand the function $f(x)$, we project $(\bar{b}_n, f(x))$ as in the equation (1), hence $f(x) = \sum_{n=0}^{\infty} f_n \bar{b}_n(x)$, with $f_n = (\bar{b}_n, f)$, see Eq. (2). Thus

$$f_n = \int_{-S}^{S} dx\, \sqrt{\frac{1}{S}}\cos(\pi nx/S) f(x) = \int_{-S/4}^{S/4} dx\, \sqrt{\frac{1}{S}}\cos(\pi nx/S) \times \underbrace{1}_{=f}$$

$$= \sqrt{\frac{1}{S}}\left(\frac{S}{\pi n}\right)\sin(\pi nx/S)\Bigg|_{-S/4}^{S/4} = 2\sqrt{S}\frac{\sin(n\pi/4)}{(n\pi)}. \tag{5}$$

For $n = 0$ we separately find $f_0 = \int_{-S/4}^{S/4} dx\, 1/\sqrt{2S} = \sqrt{S/8}$. With these exact coefficients, we can now plot the requested cumulative terms F_k, one by one e.g. the graph below.



(2) Time-dependence of a quantum harmonic oscillator:

(a) Write down the TDSE for a particle of mass m in a harmonic potential, such that the particle would classically oscillate with frequency ω [1 pt]

(b) Show that

$$\Psi(x,t) = \frac{1}{(\sigma^2\pi)^{1/4}} e^{-i\frac{\omega t}{2}} e^{-\frac{[x-x_0\cos(\omega t)]^2}{2\sigma^2}} e^{-i\left(\frac{x_0}{\sigma}\sin(\omega t)\right)\left(\frac{x-x_0\cos(\omega t)/\sqrt{2}}{\sigma}\right)} \tag{6}$$

solves that TDSE. [5 pts]

(c) Find the expectation value of the position, expectation value of the momentum, and discuss their time evolution in terms of Ehrenfest's theorem [2 pts].

(d) Finally find the probability density for this particle as a function of time, and discuss its physical meaning [2 pts].

Solution: 2(a) - For the harmonic potential $V = \frac{1}{2}m\omega^2x^2$. *Thus the Hamiltonian* $\hat{H} = \hat{T} + \hat{V} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2$. *Thus, the TDSE is given as*

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2$$

2(b) - Plugging the given ansatz of ψ *into the equation above, it can be proven that it solves the TDSE. First calculate the LHS, either by Mathematica or by hand*

$$\Psi(x,t) = \frac{1}{(\sigma^2\pi)^{1/4}} e^{-i\frac{\omega t}{2}} e^{-\frac{[x-x_0\cos(\omega t)]^2}{2\sigma^2}} e^{-i\left(\frac{x_0}{\sigma}\sin(\omega t)\right)\left(\frac{x-x_0\cos(\omega t)/\sqrt{2}}{\sigma}\right)} \tag{7}$$

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{(\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x-x_0\cos(t\omega))^2}{2\sigma^2} - \frac{ix_0\sin(t\omega)\left(x-\frac{1}{2}x_0\cos(t\omega)\right)}{\sigma^2} - \frac{it\omega}{2}\right) \\ \times i\hbar\left(-\frac{ix_0\omega\cos(t\omega)\left(x-\frac{1}{2}x_0\cos(t\omega)\right)}{\sigma^2} - \frac{x_0\omega\sin(t\omega)(x-x_0\cos(t\omega))}{\sigma^2} \right. \\ \left. - \frac{ix_0^2\omega\sin^2(t\omega)}{2\sigma^2} - \frac{i\omega}{2}\right) \end{aligned} \tag{8}$$

Simplifying further

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{i\hbar}{(\pi\sigma^2)^{1/4}}\exp\left(-\frac{(x-x_0\cos(t\omega))^2}{2\sigma^2} - \frac{ix_0\sin(t\omega)\left(x-\frac{1}{2}x_0\cos(t\omega)\right)}{\sigma^2} - \frac{it\omega}{2}\right)$$
$$\left(\frac{\omega(\sin(t\omega)+i\cos(t\omega))\left(-i\left(\sigma^2+x_0^2\right)\sin(t\omega)+\left(x_0^2-\sigma^2\right)\cos(t\omega)-2xx_0\right)}{2\sigma^2}\right) \tag{9}$$

While the RHS of TDSE is calculated as

$$\hat{H}\psi = \frac{mx^2\omega^2\exp\left(-\frac{(x-x_0\cos(t\omega))^2}{2\sigma^2} - \frac{ix_0\sin(t\omega)\left(x-\frac{1}{2}x_0\cos(t\omega)\right)}{\sigma^2} - \frac{it\omega}{2}\right)}{2(\pi\sigma^2)^{1/4}}$$
$$-\ \frac{\hbar^2}{2\sqrt[4]{\pi}m\sqrt[4]{\sigma^2}}\left[-\frac{\exp\left(-\frac{(x-x_0\cos(t\omega))^2}{2\sigma^2} - \frac{ix_0\sin(t\omega)\left(x-\frac{1}{2}x_0\cos(t\omega)\right)}{\sigma^2} - \frac{it\omega}{2}\right)}{\sigma^2}\right. \tag{10}$$
$$+\ \left(-\frac{x-x_0\cos(t\omega)}{\sigma^2} - \frac{ix_0\sin(t\omega)}{\sigma^2}\right)^2\times$$
$$\left.\exp\left(-\frac{(x-x_0\cos(t\omega))^2}{2\sigma^2} - \frac{ix_0\sin(t\omega)\left(x-\frac{1}{2}x_0\cos(t\omega)\right)}{\sigma^2} - \frac{it\omega}{2}\right)\right]$$

upon simplifying further

$$\hat{H}\psi = \frac{1}{(\pi\sigma^2)^{1/4}}\exp\left(-\frac{(x-x_0\cos(t\omega))^2}{2\sigma^2} - \frac{ix_0\sin(t\omega)\left(x-\frac{1}{2}x_0\cos(t\omega)\right)}{\sigma^2} - \frac{it\omega}{2}\right)$$
$$\left(\frac{mx^2\omega^2}{2} - \frac{\hbar^2}{2m}\left[-\frac{1}{\sigma^2} + \left(-\frac{x-x_0\cos(t\omega)}{\sigma^2} - \frac{ix_0\sin(t\omega)}{\sigma^2}\right)^2\right]\right) \tag{11}$$

To verify that the eqn (9) and eqn (11) are equal you could use the mathematicalcommand ***Fullsimplify[LHS-RHS])*** *and see that it gives zero. Thus, eqn (7) is the solution of the TDSE in 2(a).*
2(c) The expectation value of position $\langle\hat{x}\rangle$ *and momentum* $\langle\hat{p}\rangle$ *are calulcated as.*

$$\langle\hat{x}\rangle = \int_{-\infty}^{\infty}\psi^\dagger x\psi dx$$
$$\langle\hat{x}\rangle = \int_{-\infty}^{\infty}\frac{1}{(\sigma^2\pi)^{1/2}}xe^{-\frac{[x-x_0\cos(\omega t)]^2}{\sigma^2}}dx$$
$$\langle\hat{x}\rangle = x_0cos(\omega t)$$

$$\langle \hat{p} \rangle = \int_{-\infty}^{\infty} \psi^{\dagger} - i\hbar \frac{\partial}{\partial x} \psi dx$$

$$\langle \hat{p} \rangle = -i\hbar \int_{-\infty}^{\infty} dx \frac{1}{(\sigma^2 \pi)^{1/4}} e^{i\frac{\omega t}{2}} e^{-\frac{[x - x_0 \cos(\omega t)]^2}{2\sigma^2}} e^{i\left(\frac{x_0}{\sigma} \sin(\omega t)\right)\left(\frac{x - x_0 \cos(\omega t)/\sqrt{2}}{\sigma}\right)} \times$$

$$\frac{\left(-\frac{x - x_0 \cos(t\omega)}{\sigma^2} - \frac{i x_0 \sin(t\omega)}{\sigma^2}\right) e^{-\frac{(x - x_0 \cos(t\omega))^2}{2\sigma^2}} e^{-\frac{i x_0 \sin(t\omega)\left(x - \frac{1}{2} x_0 \cos(t\omega)\right)}{\sigma^2}} e^{-\frac{it\omega}{2}}}{(\pi\sigma^2)^{1/4}}$$

$$\langle \hat{p} \rangle = -i\hbar \int_{-\infty}^{\infty} dx \frac{1}{(\pi\sigma^2)^{1/2}} \left(-\frac{x - x_0 \cos(t\omega)}{\sigma^2} - \frac{i x_0 \sin(t\omega)}{\sigma^2}\right) e^{\frac{-[x - x_0 cos(\omega t)]^2}{\sigma^2}}$$

$$\langle \hat{p} \rangle = -i\hbar(0 - \frac{i x_0 sin(\omega t)}{\sigma^2})$$

$$\langle \hat{p} \rangle = -\frac{\hbar x_0 sin(\omega t)}{\sigma^2}$$

The time evolution of the expectation value of position is related to the expectation value of momentum by Ehrenfest theorem as $\frac{d\langle \hat{x} \rangle}{dt} = \frac{\langle p \rangle}{m}$*, see Eq. (1.44) of the lecture*

$$\frac{d\langle \hat{x} \rangle}{dt} = -x_0 \omega sin(\omega t)$$
$$\frac{\langle p \rangle}{m} = \frac{\hbar}{m\sigma^2} x_0 sin(\omega t)$$
$$\text{Where } \sigma = \sqrt{\frac{\hbar}{m\omega}}$$

Similary the time evolution of the expectation of the momenta is given by $\frac{d\langle \hat{p} \rangle}{dt} = -\langle \frac{dV}{dx} \rangle$

$$\frac{d\langle \hat{p} \rangle}{dt} = \frac{-\hbar x_0 \omega cos(\omega t)}{\sigma^2}$$
$$\frac{d\langle \hat{p} \rangle}{dt} = -m\omega^2 \langle \hat{x} \rangle$$
$$-\langle \frac{dV}{dx} \rangle = -m\omega^2 x$$

Observe that the Ehrenfest theorem is clearly valid for the evolution in terms of the expectation values of the position and momenta. However, it does not account for the fluctuations in the Wave-fucntion.
2(d) - The probability density of the particle is calculated as

$$|\psi^2(t,x)| = \psi^{\dagger}(t,x)\psi(t,x)$$
$$|\psi^2(t,x)| = \frac{1}{(\pi\sigma^2)^{1/2}} e^{\frac{-[x - x_0 cos(\omega t)]^2}{\sigma^2}}$$

The proability density is a Gassian around the centre $x_0 cos(\omega t)$*. Thus the Gaussian and in turn the particle will keep on oscillating with it's centre between* $x_0 and - x_0$*.*

(3) Heisenberg's uncertainty principle: **[10pts]** Show that the wavefunction Eq. (6)

of question 2 satisfies Heisenbergs uncertainty principle at all times. How do uncertainties change with time?

Solution: The position and momenta uncertainities are calculated as

$$\sigma_x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$$
$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} \psi^\dagger x^2 \psi dx$$
$$\langle \hat{x}^2 \rangle = \frac{\sigma^2 + 2x_0^2 cos(\omega t)^2}{2}$$
$$\langle \hat{x} \rangle^2 = x_0^2 cos(\omega t)^2$$
$$\therefore \sigma_x = \sqrt{\frac{\sigma^2}{2}}$$

$$\sigma_p = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}$$
$$\langle \hat{p}^2 \rangle = \int_{-\infty}^{\infty} \psi^\dagger (-\hbar^2) \frac{d^2}{dx^2} \psi \, dx$$
$$\langle \hat{p}^2 \rangle = (\hbar^2) \frac{\sigma(\sigma^2 + 2x_0^2 \sin(t\omega)^2)}{2\sqrt{\frac{1}{\sigma^2}}\sigma^6}$$
$$\langle \hat{p}^2 \rangle = \frac{\hbar^2(\sigma^2 + 2x_0^2 \sin(t\omega)^2)}{2\sigma^4}$$
$$\langle \hat{p} \rangle^2 = \frac{\hbar^2 x_0^2 \sin(t\omega)^2}{\sigma^4}$$
$$\sigma_p = \sqrt{\frac{\hbar^2(\sigma^2 + 2x_0^2 \sin(t\omega)^2)}{2\sigma^4} - \frac{\hbar^2 x_0^2 \sin(t\omega)^2}{\sigma^4}}$$
$$\sigma_p = \sqrt{\frac{\hbar^2}{2\sigma^2}}$$

$$\therefore \sigma_x \sigma_p = \frac{\hbar}{2}$$

Thus Heisenberg's uncertainty principle (Eq. (1.47) of the lecture) is satisfied. Observe that the uncertainties are independent of time.

(4) Particle bouncing in an infinite square well: Consider a particle of mass m in the infinite square well potential discussed in section 2.2.1. of the lecture. At time $t = 0$, let its wavefunction be

$$\Psi(x, t = 0) = \sqrt{\frac{1}{a}} \left(\sin\left(\pi x/a\right) - \sin\left(2\pi x/a\right)\right). \tag{12}$$

(4a) Find the wavefunction at any later time $t > 0$. [2 pts]

Solution: From Eq.(2.17) and Eq.(2.18) of the lecture, we have the Energy eigenvalues and eigenstate.

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \text{ for } n = 1, 2, 3, \ldots \tag{13}$$

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right). \tag{14}$$

using the eigenstates, the wavefunction can be written as $\Psi(x, t = 0) = 1/\sqrt{2}(\phi_1(x) - \phi_2(x))$

Now, using Eq. (1.71) of the lecture, the wavefunction at a later time t will be

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2}}\left(e^{-i\frac{E_1 t}{\hbar}}\phi_1(x) - e^{-i\frac{E_2 t}{\hbar}}\phi_2(x)\right).\\ \Psi(x,t) &= \frac{1}{\sqrt{2}}e^{-i\frac{E_1 t}{\hbar}}\left(\phi_1(x) - e^{-i\frac{(E_2-E_1)t}{\hbar}}\phi_2(x)\right)\end{aligned} \tag{15}$$

(4b) From that calculate the probabilities that the particles is in the left side of the well (at $0 < x < a/2$) or the right side ($a/2 < x < a$) [3 pts]

Solution: The probability density is given by

$$\begin{aligned}|\Psi(x,t)|^2 &= \frac{1}{2}\left(|\phi_1(x)|^2 + |\phi_2(x)|^2 - e^{-i\frac{(E_2-E_1)t}{\hbar}}\phi_1(x)\phi_2(x) - e^{i\frac{(E_2-E_1)t}{\hbar}}\phi_1(x)\phi_2(x)\right)\\ &= \frac{1}{2}\left(|\phi_1(x)|^2 + |\phi_2(x)|^2 - 2\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\phi_1(x)\phi_2(x)\right).\end{aligned} \tag{16}$$

Now the probability of being on the left side of the well (at $0 < x < a/2$) is given by,

$$\begin{aligned}P_L(t) &= \int_0^{a/2} dx\, |\Psi(x,t)|^2\\ &= \frac{1}{2}\left(\int_0^{a/2} dx\, |\phi_1(x)|^2 + \int_0^{a/2} dx\, |\phi_2(x)|^2 - \int_0^{a/2} dx\, 2\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\phi_1(x)\phi_2(x)\right).\end{aligned}$$

here the term $\int_0^{a/2} dx\, |\phi_1(x)|^2$ is the probability to find the particle in the left half if the particle is in the first eigenstate and it is equal to $1/2$ and same for all eigenstates. also $\int_0^{a/2} dx\, \phi_1(x)\phi_2(x) = 4/3\pi$ So we get

$$P_L(t) = \frac{1}{2}\left(1 - \frac{8}{3\pi}\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\right). \tag{17}$$

Now the probability of finding the particle in the right half is

$$P_R(t) = 1 - P_L(t) = \frac{1}{2}\left(1 + \frac{8}{3\pi}\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\right). \tag{18}$$

(4c) Calculate the probability current at $x = a/2$ as a function of time, and relate your finding with the answer to (4b). [3 pts]

Solution: The form of probability current is given in Eq. (1.53) lecture notes.

$$J(x,t) = \frac{\hbar}{m}\mathfrak{Im}\left[\Psi^*\frac{\partial \Psi}{\partial x}\right] \tag{19}$$

$$\begin{aligned}\frac{\partial \Psi}{\partial x} =& \frac{\partial}{\partial x}\left[\sqrt{\frac{1}{a}}\left(\sin\left(\pi x/a\right) - e^{-i\frac{(E_2-E_1)t}{\hbar}}\sin\left(2\pi x/a\right)\right)\right].\\ =&\sqrt{\frac{1}{a}}\frac{\pi}{a}\left(\cos\left(\pi x/a\right) - 2e^{-i\frac{(E_2-E_1)t}{\hbar}}\cos\left(2\pi x/a\right)\right)\end{aligned} \tag{20}$$

using this

$$\begin{aligned}\Psi^*\frac{\partial \Psi}{\partial x} &= \frac{\pi}{a^2}\left(\cos\left(\pi x/a\right) - 2e^{-i\frac{(E_2-E_1)t}{\hbar}}\cos\left(2\pi x/a\right)\right)\left(\sin\left(\pi x/a\right) - e^{i\frac{(E_2-E_1)t}{\hbar}}\sin\left(2\pi x/a\right)\right)\\ &= \frac{\pi}{a^2}\left(\cos\left(\pi x/a\right)\sin\left(\pi x/a\right) + 2\cos\left(2\pi x/a\right)\sin\left(2\pi x/a\right)\right.\\ &\left.-2e^{-i\frac{(E_2-E_1)t}{\hbar}}\cos\left(2\pi x/a\right)\sin\left(\pi x/a\right) - e^{i\frac{(E_2-E_1)t}{\hbar}}\cos\left(\pi x/a\right)\sin\left(2\pi x/a\right)\right)\end{aligned} \tag{21}$$

Taking the imaginary part of equation (21) and putting it in equation (19) we have

$$J(x,t) =\frac{\hbar\pi}{ma^2}\left(2\sin\frac{(E_2-E_1)t}{\hbar}\cos\left(2\pi x/a\right)\sin\left(\pi x/a\right) - \sin\frac{(E_2-E_1)t}{\hbar}\cos\left(\pi x/a\right)\sin\left(2\pi x/a\right)\right) \tag{22}$$

The probability current at $x = a/2$ is given by

$$J(a/2,t) = \frac{\hbar\pi}{ma^2}\left(-2\sin\frac{(E_2-E_1)t}{\hbar}\right) \tag{23}$$

Now the current is also the rate of change of probability so we must have

$$\begin{aligned}J(t) = \frac{dP_L(t)}{dt} &= \frac{4}{3\pi}\sin\left(\frac{(E_2-E_1)t}{\hbar}\right)\frac{E_2-E_1}{\hbar}\\ &= \frac{4}{3\pi}\sin\left(\frac{(E_2-E_1)t}{\hbar}\right)\frac{3\pi^2\hbar^2}{2ma^2\hbar}\end{aligned} \tag{24}$$

$$= \frac{\hbar\pi}{ma^2}\left(-2\sin\frac{(E_2-E_1)t}{\hbar}\right) \tag{25}$$

(4d) What is the probability to measure the particle to have energy $E = \pi^2\hbar^2/(2ma^2)$ at $t = 0$? What about later times $t > 0$? [1 pt]

Solution : The wave function at time $t = 0$ is given by

$$\Psi(x, t = 0) = 1/\sqrt{2}(\phi_1(x) - \phi_2(x)) \tag{26}$$

from this, the probability to measure $E = \pi^2\hbar^2/(2ma^2)$ *is square of the coefficient of* $\phi_1(x)$ *i.e.,* $1/2$. *Later, the wave function at time* t *is given by*

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2}}\left(e^{-i\frac{E_1 t}{\hbar}}\phi_1(x) - e^{-i\frac{E_2 t}{\hbar}}\phi_2(x)\right) \tag{27}$$

Since the coefficient is modified by a phase only, the probability of measuring $E = \pi^2\hbar^2/(2ma^2)$ *is still the same, i.e.,* $1/2$.

(4f) Suppose we have measured the energy to be $E = \pi^2\hbar^2/(2ma^2)$ *at time* $t_m = (2ma^2)/(\pi\hbar)$ *and subsequently measure it again at* $t = 2t_m$. *What is the probability to find the same value* E *again? [1 pt]*

Solution : When we measure the energy to be $E = \pi^2\hbar^2/(2ma^2)$ *at time* $t_m = (2ma^2)/(\pi\hbar)$ *our states collapse to the eigenstate* $\phi_1(x)$. *i.e., at time* t_m

$$\Psi(x, t_m) = \phi_1(x) \tag{28}$$

when it evolved from here, the state at the time t *will be*

$$\Psi(x, t) = e^{-i\frac{E_1(t-t_m)}{\hbar}}\phi_1(x) \tag{29}$$

The state at $t = 2t_m$ *is*

$$\begin{aligned}\Psi(x, t) =& e^{-i\frac{\pi^2\hbar}{2ma^2}\frac{2ma^2}{\pi\hbar}}\phi_1(x)\\ =& e^{-i\pi}\phi_1(x)\end{aligned} \tag{30}$$

the probability of finding the same value E *again is one as the state is the same up to a phase factor.*