PHY 303, I-Semester 2022/23, Assignment 1

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Due-date: Sun 13th Aug 2023 23:55

(1) Relativistically accelerated electrons: [**10pts**] Free electrons in a plasma can be accelerated to relativistic velocities using very strong lasers. Suppose, in a certain experiment, the probability distribution of one cartesian component of the momentum after interaction with the laser is given by the function drawn below, where $a < b < c$ are three values of momentum (all $n \times m_e c$, with $1.5 < n < 10.8$, where m_e is the electron mass and c the speed of light[1]).

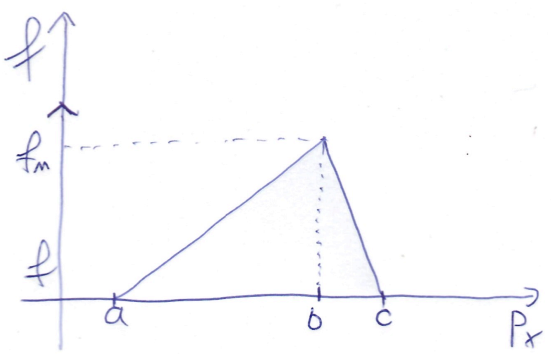

Figure 1: Continuous probability distribution $f(p_x)$ of a certain cartesian component p_x of electron momentum after some relativistic laser plasma interaction.

(1a) Setup a function for the probability distribution $f_{a,b,c}(p_x)$ with $a < b < c$ matching the above drawing, and determine f_m such that this distribution is correctly normalised. Then find the mean, most-likely and median momentum that we would measure. Also find the standard deviation. For all four quantities and the parameters a, b, c, discuss their meaning/implication for measurements. Discuss why they take the values that you found, for this specific distribution. Check the sanity of all your answers by looking at some distributions where (some of) the answers are obvious beforehand, such as a symmetric distribution, or an arbitrarily narrow distribution.

(1b) Now suppose some complicated laser-plasma interaction generates a bi-modal distribution of the form

$$g(p_x) = p_0 f_{a,b,c}(p_x) + p_1 f_{d,e,f}(p_x), \tag{1}$$

with $a < b < c < d < e < f$. What relation do p_0 and p_1 have to fulfill for g to be properly normalised? Also find the mean momentum and standard deviation again for the distribution g.

(1c) Consider the special case where $|d - c| \gg |c - a|$, $|f - d|$ for $p_0 = p_1$. Which combination of parameters now dominantly sets the standard deviation and why does that make sense? Will you ever measure a momentum close to the mean momentum?

[1]You shall find that most of these details do not matter for the question

(2) Matrix diagonalisation: [10pts] Find the eigenvalues and eigenvectors of the following matrices. Normalize the eigenvectors. Then explicitly verify the construction in Eq. (1.16). For the real matrix, how and why can you interpret the diagonalisation as a basis change in $\mathbb{R}^3$? For the real matrix, find the complete eigenspaces, i.e. the set of *all* vectors that are an eigenvector if they are not normalised. *Please provide a manual solution, not using a computer. Then verify your solution with mathematica and document that.*

$$\underline{\underline{M}}_1 = \frac{1}{3}\begin{bmatrix} 8 & -1 & -1 \\ -1 & \frac{13}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{13}{2} \end{bmatrix}, \quad \underline{\underline{M}}_2 = \frac{1}{3}\begin{bmatrix} 7 & 1 & 2i \\ 1 & 4 & -i \\ -2i & i & 7 \end{bmatrix}. \tag{2}$$

(3) Wavefunction evolution and collapse: [8pts] On the teams page I provide a recording of my discussion of example 1, also watch the provided movie `TDSE_demo_overbarrier.gif`, and optionally my lecture notes for PHY106 (week8 , page 20). Let us assume a quantum mechanical particle of mass m is described by the wavefunction $\Psi(x, 0)$ drawn below, and subject to a potential energy $V(x)$ as shown.

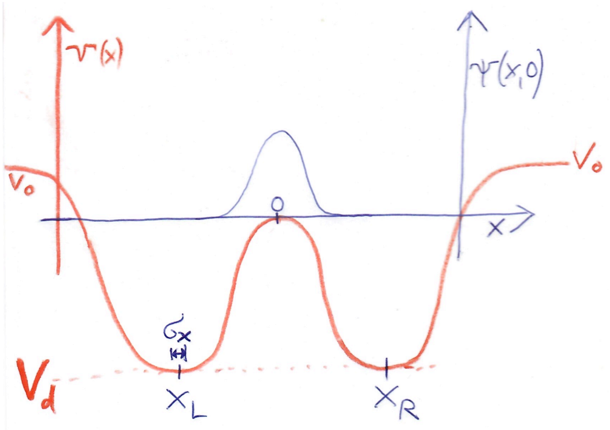

Figure 2: Initial wavefunction $\Psi(x, t = 0)$ (blue) of a particle subject to the potential energy $V(x)$ shown in orange. The potential asymptotically becomes $V(x) \to V_0$ at large $|x|$ and two minima with energy V_d at $x = x_L, x_R$.

(i) Based on this, provide your best guess how the wavefunction $\Psi(x, t)$ evolves at later times $t > 0$. Explain or justify your guess. Also discuss how particles treated in classical mechanics, that start in a comparable initial configuration, would be expected to evolve. At which time(s) do you expect the quantum and classical answers to possibly strongly differ, how and why? *Hint: To most of these questions, you cannot yet know the answer rigorously. Trust your intuition and what you learnt in PHY106 for now, later in the course we shall check if your guess was correct.*

(ii) Now suppose, at some time $t > 0$, we measure the position of the particle near $x \approx x_L$ up to an accuracy σ_x as shown in the figure. (Let us assume $|\Psi(x, t)|^2 > 0$ is significant for $x \approx x_L$ at this time). What is the wavefunction of the particle immediately after that measurement at $t = t_{\text{meas}}$?

(iii) **Bonus question:**[2] Assume further that $\sigma_x \ll \hbar^2/[8m(V_d + V_0)]$. What is the expected future behaviour of that particle at times $t \gg t_{\text{meas}}$? (or one dominant behaviour? *Hint: If you learnt about the Fouriertransform/ Momentum distribution of a Gaussian wavefunction in PHY106 attempt this. Otherwise, revisit it after week 4 in this lecture.)*

(4) Computational question, discretised function vector spaces [12 pts]: Consider the set of functions on the interval $x \in [0, a]$ given by $\{f_n\} = \{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)\}$ for $n = 1, 2, 3, \cdots$. The script `Assignment1_draft.nb` is set up to represent any function on a discretisation of that spatial interval into N_{pts} equidistant points: $x_k = k\left(\frac{a}{N_{\text{pts}}}\right)$ for $k = 0, 1, 2, \cdots N_{\text{pts}} - 1$ and visualise these.

(4a) Use this to visualise of a few of these functions and discuss how you can relate them to a N_{pts} component vector. Demonstrate that these vectors are all mutually orthogonal. For a given N_{pts}, what seems to be the maximum n for which f_n can still be reasonably visualised?

(4b) Pick an arbitrary function g of your taste, that satisfies $g(0) = g(a) = 0$ (or ≈ 0), and visualise this in the same way. Then express it in terms of the functions above as $g = \sum_{k=0}^{k_{\text{max}}} c_k \bar{f}_k$, and plot this cumulative sum for a couple of choices of k_{max}. Here $\bar{f}_k$ are the discretised function vectors from part (a), but normalised to one. Discuss what you see.

(4c) Calculate $h_n = f_n'''$ first analytically, the third spatial derivative of f_n. Then set up a matrix (operator), which you can multiply with the column vectors from (a) such that they calculate that third derivative. In the end show through a plot explicitly that your construction does its job as expected. What does it tell us about the relation between operators and matrices?

[2]This means you loose no marks when you don't do it.