Phys 637, I-Semester 2022/23, Tutorial 8 solution

Stage 1 *(Quantum optical master equation)* Revisit the master equation (4.87).

(i) Describe all the *physical* ingredients for the scenario it describes.
Solution: There is a coherent coupling between levels $|\,g\,\rangle$ *and* $|\,e\,\rangle$ *e.g. provided by a laser. The strength of that coupling is given by* Ω*, related to the light intensity, while the frequency of the laser enters the detuning* Δ *(which is the difference between transition frequency and laser frequency). In addition there may be incoherent black-body radiation at the transition frequency, which causes stimulated emission or can be absorbed and then drives an incoherent excitation. Finally our treatment has incorporated spontaneous decay as well.*

(ii) What does each of the terms on the rhs of Eq. (4.89) do/describe? When are they zero/non-zero, large/small?
Solution: (on demand only)

(iii) When atoms are in $|\,g\,\rangle$ they can change into $|\,e\,\rangle$ either using the Rabi coupling from the Laser or via absorbing a photon from the black-body radiation environment. What is the practical difference in how these processes enter the Master equation (4.89)? What is the physical difference?
Solution: The mathematical difference is that the environmental effects couple populations directly, e.g. in $\rho_{gg} = \cdots + \gamma\rho_{ee}$*. This is akin to a classical rate equation. In contrast the laser coupling is exclusively via the coherences (cannot proceed without). As a result the former does not give rise to oscillations (only relaxation), while the latter does. Physically, the difference is the coherence of the light fields in question: The laser has a well defined phase of oscillation at all times (if you revisit the PHY402 origin of Eq. (4.88), one assumes something like* $E(x,t) = E_0\cos(\omega t - kx)$ *for its electric field, which is a perfectly coherent monochromatic wave. In contrast the thermal environment due to black-body photons has to be imagined as an incoherent mixture of photons with lots of different phases (see e.g. discussion in section 1.2. and references therein).*

Stage 2 *(Adiabatic elimination)* Consider a two level atom as in example 37, Eq. (4.88), but without the environment. We slightly redefine the energies of the states, so that the Hamiltonian is

$$\hat{H}_{\mathcal{S}} = \frac{\Omega}{2}(|\,e\,\rangle\langle\,g\,| + |\,g\,\rangle\langle\,e\,|) - \Delta|\,e\,\rangle\langle\,e\,|. \tag{1}$$

(i) Write down the Schrödinger equation for state amplitudes in $|\,\Psi(t)\,\rangle = c_g(t)|\,g\,\rangle + c_e(t)|\,e\,\rangle$.

Solution: We apply $\hat{H}_{\mathcal{S}}$ to $|\,\Psi(t)\,\rangle$ and then project the TDSE onto $|\,g\,\rangle$ and $|\,e\,\rangle$ to reach

$$i\hbar\frac{\partial}{\partial t}c_g(t) = \frac{\Omega}{2}c_e(t), \tag{2}$$
$$i\hbar\frac{\partial}{\partial t}c_e(t) = \frac{\Omega}{2}c_g(t) - \Delta c_e(t). \tag{3}$$
$$\tag{4}$$

(ii) Let us assume that $\Delta \gg \Omega$. Then generalize (in fact it is simpler) the technique of adiabatic elimination to a wavefunction, to get rid of amplitudes c_e and find an equation for c_g only.
Solution: Due to $\Delta \gg \Omega$ we can use the argument required for adiabatic elimination that $\frac{\partial}{\partial t}c_e(t)$ is a very rapidly oscillating complex number and thus averages to zero. Solving that averaged equation gives:

$$0 = \frac{\Omega}{2}c_g(t) - \Delta c_e(t) \Rightarrow$$
$$c_e(t) = \frac{\Omega}{2\Delta}c_g(t). \tag{5}$$

Inserting this into Eq. 2 gives:

$$i\hbar\frac{\partial}{\partial t}c_g(t) = \underbrace{\left(\frac{\Omega}{2\Delta}\right)^2}_{=\alpha^2}\Delta c_g(t), \tag{6}$$

The way the RHS looks now $\bar{E} = \alpha^2\Delta$ represents an energy shift of the ground-state. This is called light shift.

(iii) Now let us assume the laser intensity is spatially dependent $\Omega = \Omega(\mathbf{x})$. Discuss the meaning of the term you found above. What can we do with it?
Solution: In that case we have $\bar{E}(x) = \frac{\Omega(x)^2}{4}\Delta$, which is a spatially dependent potential that depends on the light intensity (through $\Omega(x)$) and light frequency (through Δ). It can be used for optical trapping.

Stage 3 *(Steady states)* Derive the results in example 37 for the steady state of an atom under laser drive and spontaneous decay.
Solution: (on demand only)