Phys 637, I-Semester 2022/23, Tutorial 7 solution

We suggest to do "Stages" in the order below, feel free to change that as per your interests. Discuss first on your table within your team, then with neighboring tables.

Stage 1 *(Lindblad Masterequation)* Consider a three-level system with Hamiltonian

$$\hat{H} = E_0|\,0\,\rangle\langle\,0\,| + E_1|\,1\,\rangle\langle\,1\,| + E_2|\,2\,\rangle\langle\,2\,|, \tag{1}$$

and Lindblad operator $\hat{L} = \sqrt{\kappa}|\,2\,\rangle\langle\,2\,|$.

(i) Derive the Lindblad Masterequation for that problem.
Solution: We have to evaluate the RHS of Eq. (4.25), and then take matrix elements $\langle\,k\,|\cdots|\,k'\,\rangle$ on both sides to extract the evolution equation $\dot{\rho}_{kk'} = \cdots$. Equivalently, for this simple 3×3 problem, we can just write everything in matrix form:

$$\hat{\rho}(t) = \begin{pmatrix} \rho_{00}(t) & \rho_{01}(t) & \rho_{02}(t) \\ \rho_{10}(t) & \rho_{11}(t) & \rho_{12}(t) \\ \rho_{20}(t) & \rho_{21}(t) & \rho_{22}(t) \end{pmatrix}, \qquad \hat{H} = \begin{pmatrix} E_0 & 0 & 0 \\ 0 & E_1 & 0 \\ 0 & 0 & E_2 \end{pmatrix}, \qquad \hat{L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{\kappa} \end{pmatrix}. \tag{2}$$

Evaluating the RHS then amounts to just a bunch of matrix multiplications, and we find (e.g. using mathematica):

$$\begin{pmatrix} \dot{\rho}_{00}(t) & \dot{\rho}_{01}(t) & \dot{\rho}_{02}(t) \\ \dot{\rho}_{10}(t) & \dot{\rho}_{11}(t) & \dot{\rho}_{12}(t) \\ \dot{\rho}_{20}(t) & \dot{\rho}_{21}(t) & \dot{\rho}_{22}(t) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -i\frac{(E_0-E_1)}{\hbar}\rho_{01}(t) & [-i[\frac{(E_0-E_2)}{\hbar} - i\frac{\kappa}{2}]\rho_{02}(t) \\ i\frac{(E_0-E_1)}{\hbar}\rho_{10}(t) & 0 & [-i[\frac{(E_1-E_2)}{\hbar} - i\frac{\kappa}{2}]\rho_{12}(t) \\ i[\frac{(E_0-E_2)}{\hbar} + i\frac{\kappa}{2}]\rho_{20}(t) & [i\frac{(E_1-E_2)}{\hbar} + i\frac{\kappa}{2}]\rho_{21}(t) & 0 \end{pmatrix}. \tag{3}$$

(ii) Compare the expected time-evolution for initial states $|\,\phi_a\,\rangle = (|\,0\,\rangle + |\,1\,\rangle)/\sqrt{2}$ and $|\,\phi_b\,\rangle = (|\,0\,\rangle + |\,2\,\rangle)/\sqrt{2}$. How do you interpret this? What parameter(s) govern(s) the time-scale for decoherence? Which don't?
Solution: Luckily (and atypically for a Lindblad equation), the time-derivatives for all matrix elements in (3) de-couple, and the resultant equations can all easily be solved. We have $\rho_{kk}(t) = \rho_{kk}(0) = const$ for all populations and for $(k\ell) = (02)$ or (12) we have

$$\rho_{k\ell}(t) = \rho_{k\ell}(0)e^{-\frac{\kappa}{2}t}e^{-i\frac{(E_k-E_\ell)}{\hbar}t}. \tag{4}$$

while

$$\rho_{01}(t) = \rho_{01}(0)e^{-i\frac{(E_0-E_1)}{\hbar}t}. \tag{5}$$

The initial wavefunctions given provide us with all initial density matrix elements. For the case $|\phi_a\rangle$, $\rho_{01}(0) = 1/2$ *is the only initially non-zero coherence, it remains nonzero, with a complex exponential phase factor oscillating at the energy difference between* $|0\rangle$ *and* $|1\rangle$. *In contrast, when starting from* $|\phi_b\rangle$, *the initially non-zero* $\rho_{02}(0) = 1/2$ *is exponentially damped on a time-scale* $\tau_{decoh} \sim 1/\kappa$.

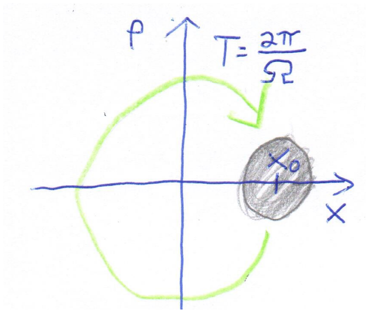
Mathematically, this happens because the Lindblad operator did not contain $|0\rangle$ *or* $|1\rangle$, *so only superpositions involving* $|2\rangle$ *dephase. Physically, this likely means that the system-environment interaction was state-dependent, such that it only affects state* 2, *shifting its energies conditional on the environment with a system operator* $\hat{S} \sim |2\rangle\langle 2|$. *We see that the only parameter controlling the decoherence timescale is* κ, *while other parameters in the problem* (E_k) *do* not *affect that.*

(iii) From the discussion in week6, which type of system environment interaction Hamiltonian is likely responsible for such a Lindblad operator and what would the operator mean physically?
Solution: See above: Physically, this likely means that the system-environment interaction was state-dependent, such that it only affects state 2, *shifting its energies conditional on the environment with a system operator* $\hat{S} \sim |2\rangle\langle 2|$.

(iv) Derive the Lindblad equation from the Born-Markov equation for the case of Hermitian $\hat{S}_\alpha$ by assuming zero memory time of the environment $\mathcal{C}_{\alpha,\beta}(\tau) = \gamma_{\alpha,\beta}\delta(\tau)$. Use the yellow box on page 76 of the lecture notes for guidance.
Solution: See yellow box on page 76

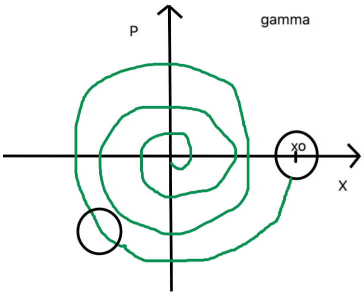
Stage 2 *(Quantum Brownian motion)* Let us consider a simpler initial state for quantum Brownian motion than in example 30 of the lecture: $\Psi(x) = \mathcal{N} e^{-\frac{(x-x_0)^2}{2\sigma^2}}$.

(i) Make a sketch of its Wigner function $W(x,p)$ on the board. Based on that discuss how $W(x,p,t)$ should evolve in time for an oscillator without an environment.
Solution: The initial Wignerfunction has a 2D Gaussian shape, indicating the initial momentum and position uncertainty. It remains in that shape, periodically encircling the origin as in the phase-space portrait of the classical harmonic oscillator.

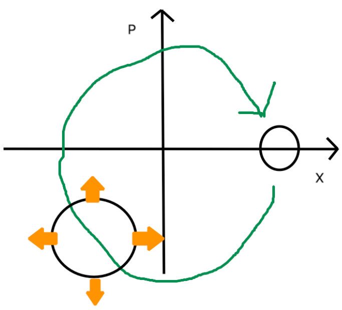


(ii) Based on the description of the terms $\sim \gamma$ and $\sim D$ in the Master equations (4.51) and (4.57), how would you expect each to affect that evolution of the Wignerfunction, separately and together? At early times and late times? Use intuition and educated guesses and board drawings, recording your ideas.

Solution: (early times) We have learnt that the term $\sim \gamma$ describes friction. Friction causes the oscillator to loose energy, hence its phase space orbit should spiral into the origin. Initially, based on that we would thus guess a behaviour as shown below:

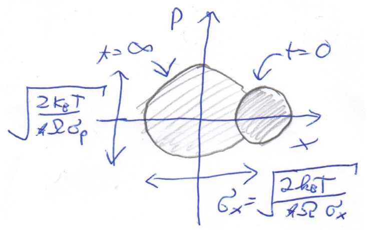


The term $\sim D$ is responsible for the loss of spatial coherence, spatial diffusion and momentum diffusion. The former leads to the loss of any negative pieces in the Wigner function (which are not there for the initial state above, but are there for the superposition of two Gaussians in example 30). Spatial diffusion extends the support of W in the x direction. Momentum diffusion will increase the momentum uncertainty, extending the support of W in the p direction. Hence W diffuses in BOTh phase space direction.



(late times) Diffusion cannot go on forever: If momentum and position

diffusion were going on forever, they would lead to arbitrarily large energies. That seems physically unreasonable (but note that a master equation DOES NOT conserve energy, since it deals with the system part of the complete universe only). Since in our QBM we are describing a harmonic oscillator in contact with an environment at temperature T, we would expect it to reach thermal equilibrium (thermalize) with that environment. At long times $t \to \infty$, the Wigner function thus should approach that of a thermal state [see Eq. (4.37)] of the central oscillator and then cease to change.



Stage 3 *(Perpetually positive density matrices)*

(i) Show that if and only if

$$\langle \Psi | \hat{\rho}_S(t) | \Psi \rangle \geq 0, \tag{6}$$

for all possible states $| \Psi \rangle$, then the populations p_n of the density matrix are ≥ 0 in any basis.

Solution: We show both directions of the equivalence separately. Let $| \Psi \rangle = \sum_n c_n | \phi_n \rangle$ be an arbitrary state expressed in the (Arbitrary) basis for which we want to show all populations to be positive, and let's target population of state k. Then we know

$$0 \leq \langle \Psi | \hat{\rho}_S(t) | \Psi \rangle = \sum_{nm} c_n^* c_m \langle \phi_n | \hat{\rho}_S(t) | \phi_m \rangle \overset{\text{[Let's choose } c_n = \delta_{nk}]}{=} \rho_{kk}. \tag{7}$$

Thus ρ_{kk} must be positive. In the reverse direction, assume we know all *populations in* all *choices of basis are positive. We want to show (6) for an arbitrary state $| \Psi \rangle$. Thus let us choose a basis that contains $| \Psi \rangle$ as one of the basis vectors, say $k = 0$. Then $\langle \Psi | \hat{\rho}_S(t) | \Psi \rangle = \rho_{kk} \geq 0$, as we had to show.*

(ii) Show that if the property Eq. (6) is true at time t, then evolution according to Kraus operators as in Eq. (3.66) of the lecture preserves the property at all later times $t' > t$.

Solution: We know that for any *state $| \Psi \rangle$*

$$\langle \Psi | \hat{\rho}_S(0) | \Psi \rangle \geq 0, \tag{8}$$

We also know from Eq. (3.66) that

$$\hat{\rho}(t)_{\mathcal{S}} = \sum_{ij} \hat{E}_{ij}(t)\hat{\rho}(0)_{\mathcal{S}}\hat{E}^{\dagger}_{ij}(t). \tag{9}$$

Inserting into (8) gives

$$\langle\, \Psi \,|\sum_{ij} \hat{E}_{ij}(t)\hat{\rho}(0)_{\mathcal{S}}\hat{E}^{\dagger}_{ij}(t)|\, \Psi \,\rangle = \sum_{ij} \underbrace{\langle\, \Psi \,|\hat{E}_{ij}(t)}_{\equiv\langle\, \phi_{ij}\,|}\hat{\rho}(0)_{\mathcal{S}}\underbrace{\hat{E}^{\dagger}_{ij}(t)|\, \Psi \,\rangle}_{\equiv|\, \phi_{ij}\,\rangle}. \tag{10}$$

Defining $\hat{E}^{\dagger}_{ij}(t)|\, \Psi \,\rangle = |\, \phi_{ij} \,\rangle$ *and using Eq. (8) for* $|\, \Psi \,\rangle \to |\, \phi_{ij} \,\rangle$, *we have that (10) is a sum of positive numbers and thus positive.*