Phys 637, II-Semester 2020/21, Tutorial 4 solution

We suggest to do "Stages" in the order below, feel free to change that as per your interests. Discuss first on your table within your team, then with neighboring tables.

Stage 1 *(Pointer states)*

(i) Consider the following System-Apparatus-Environment Hamiltonian, for a system spin, apparatus harmonic oscillator and multiple environment oscillators:

$$\mathcal{H}_{\mathcal{S}} = \Delta E \hat{\sigma}_z, \qquad \mathcal{H}_{\mathcal{A}} = \hbar\omega \hat{a}^\dagger \hat{a}, \qquad \mathcal{H}_{\mathcal{E}} = \sum_n \hbar\omega_n \hat{b}_n^\dagger \hat{b}_n,$$

$$\mathcal{H}_{\mathcal{SA}} = \underbrace{\kappa_{SA}\hat{\sigma}_y}_{=\hat{S}}(\hat{a} + \hat{a}^\dagger), \qquad \mathcal{H}_{\mathcal{AE}} = \underbrace{(\hat{a} + \hat{a}^\dagger)}_{=\hat{A}} \otimes \sum_n \eta_n(\hat{b}_n^\dagger + \hat{b}_n). \tag{1}$$

Find the pointer states (of the system, with respect to the apparatus), and the pointer states (of the apparatus, with respect to the environment).
Solution: The pointer states of the system, with respect to the apparatus are the eigenstates of the S part of $\mathcal{H}_{\mathcal{SA}}$, which is $\hat{S}$ as shown above. These are $|\pm_y\rangle = (|\uparrow\rangle \pm i|\downarrow\rangle)/\sqrt{2}$. The pointer states of the apparatus, with respect to the environment are the eigenstates of the A part of $\mathcal{H}_{\mathcal{AE}}$, which is $\hat{A}$ as shown above. We have reshuffled the sum, but we are free to do so. We know that $\hat{A} \sim \hat{q}$ where $\hat{q}$ is the position of the apparatus oscillator, hence the pointer states of the apparatus, with respect to the environment are position "eigenstates" $|q\rangle$.

Stage 2 *(Wigner function)*

(i) Revise the properties and purpose of the Wigner function that we stated in the lecture.

(ii) Based on that, without a calculation, discuss qualitatively how you think the Wigner function of the following quantum states should look like, or at least guest SOME of its features, and make a drawing (isocontours) of them in the $x - p$ phase-space:

- (a) The $n = 5$ eigenstate of the 1D quantum harmonic oscillator $\phi_5(x)$.
- (b) A plane wave e^{ikx}.
- (c) An Airy function $Ai(x)$.

Solution: We can get guidance by three features: The support (areas of nonvanishing function) in the $x - p$ plane will be mainly where we expect a classical state with the same features as the quantum state to sit. (a) In the case of the oscillator eigenstate this means at energy $E = \hbar\omega(5 + 1/2)$. We know from classical mechanics that this is an ellipse in phase space, as drawn in Fig. 1. But we also require that $P(x) = \int dp W(x,p)$ is the *position probability distribution and $P(p) = \int dx W(x,p)$ the momentum one. In oscillator eigenstate $n = 5$, both of these have many nodes, due to the Hermite polynomial. The drawing would not satisfy this requirement, so there must be lots of additional features, hence we wrote the red ?. (b) for the plane wave we know that $p = \hbar k$ exactly and x is completely unknown. So we suspect a straight line in phase space as drawn. This time there is not problem with getting the right $P(x)$ and $P(p)$, hence no question marks. (c) We know the Airy function is the eigenfunction of the particle in a linear potential $V(x) = Fx$, i.e. with a constant force. Classically, the trajectory of such a particle is $x(t) = x_0 + \frac{1}{2}\frac{F}{m}t^2$ and $p(t) = p_0 + Ft$. We can use the latter to eliminate t and find $x(p) = x_0 + \frac{1}{2}\frac{(p-p_0)^2}{m}$, hence x as a function of p is a parabola. For some arbitrary choice of x_0 and p_0, this is drawn in Fig. 1. Again, we have the issue that the density in an Airy function $P(x) = |Ai(x)|^2$ is highly oscillatory, so some additional ? features have to be there.*

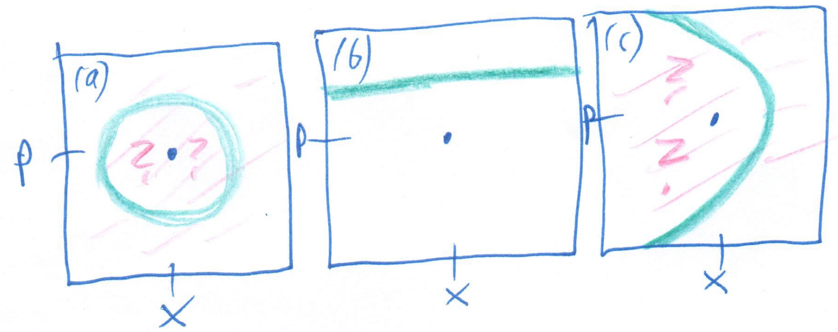

Figure 1: Reasonable guesses for the main support of Wignerfunctions with lettering as above.

(iii) (later at home) Corroborate your thoughts from the tutorial with actual calculations, or numerical plots.
Solution: We can just use the definition of the Wigner function for some examples to produce the plots shown in Fig. 2. We see that we have guessed the dominant features correctly, and the numerics now have provided the "complicated extra features", which would be harder to guess. Having plotted them, looking at the signs, and doing the integration over x and p from the figure "in our head", we can however see that these features can in principle provide us with the required node structure of e.g. position space densities.

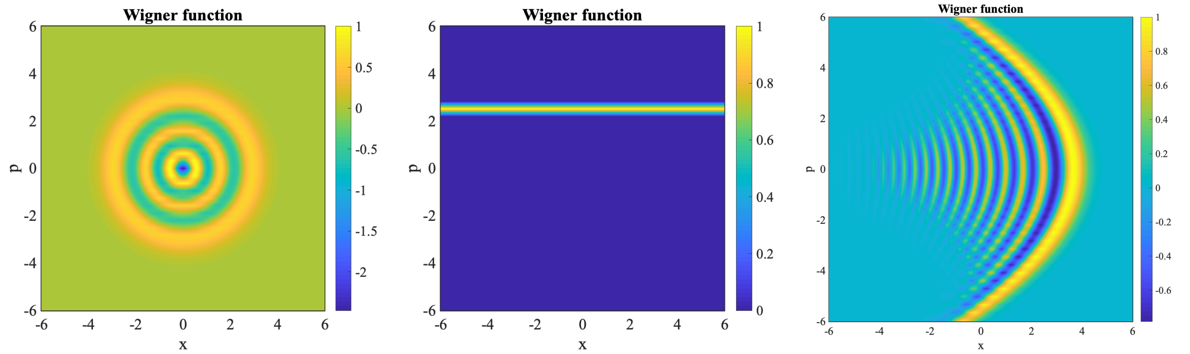

Figure 2: Actual Wigner functions from calculations, left to right in the same ordering as Fig. 1, using `Tutorial4_plot_wignerfunction.m`, showing agreement with our guess on the main features plus some harder to guess quantum interference features. For the Airy function we had to multiply an additional windowfunction for a cleanish Fourier transform.

Stage 3 *(Schmidt-decomposition)*

(i) Using the Schmidt-decomposition (3.48), obtain expressions for the reduced density matrices for system $\mathcal{A}$ and system $\mathcal{B}$. Show that these reduced density matrices are Hermitian and find their eigenvalues and eigenvectors. With that, show that the Schmidt-decomposition for any state $|\Psi\rangle$ involving systems $\mathcal{AB}$ can be found by finding the two reduced density matrices in any basis and diagonalising them.

Solution: The Schmidt-decomposition of an arbitrary pure state $|\Psi\rangle$ of the composite system $\mathcal{AB}$ is of the form:

$$|\Psi\rangle = \sum_n \lambda_n |a_n\rangle |b_n\rangle, \tag{2}$$

where $\{|a_n\rangle\}$ $[\{|b_n\rangle\}]$ are orthonormal basis vectors of the Hilbert space $\mathcal{H}_\mathcal{A}[\mathcal{H}_\mathcal{B}]$. The total density matrix is thus:

$$\begin{aligned} \rho = |\Psi\rangle\langle\Psi| &= \sum_{mn} \lambda_m \lambda_n^* |a_m\rangle|b_m\rangle\langle a_n|\langle b_n| \\ &= \sum_{mn} \lambda_m \lambda_n |a_m\rangle\langle a_n| \otimes |b_m\rangle\langle b_n| \end{aligned} \tag{3}$$

Since $|b_n\rangle$ are a basis, we can use it to perform the trace over B. The reduced density matrix for system $\mathcal{A}$ is therefore:

$$\begin{aligned} \rho_\mathcal{A} = Tr_\mathcal{B}\{\rho\} &= \sum_p \langle b_p| \left(\sum_{mn} \lambda_m \lambda_n |a_m\rangle\langle a_n| \otimes |b_m\rangle\langle b_n| \right) |b_p\rangle \\ &= \sum_p \sum_{mn} \lambda_m \lambda_n |a_m\rangle\langle a_n| \otimes \underbrace{\langle b_p|b_m\rangle}_{\delta_{pm}} \underbrace{\langle b_n|b_p\rangle}_{\delta_{np}} \\ \rho_\mathcal{A} &= \sum_p \lambda_p^2 |a_p\rangle\langle a_p|. \end{aligned} \tag{4}$$

Similarly,

$$\rho_{\mathcal{B}} = \sum_q \lambda_q^2 |b_q\rangle\langle b_q|. \tag{5}$$

Since $\lambda_p^2 = \lambda_p^{*2}$, *it is easily evident that* $\rho_{\mathcal{A}}^\dagger = \rho_{\mathcal{B}}$ *and so* $\rho_{\mathcal{A}}$ *is Hermitian. Similarly,* $\rho_{\mathcal{B}}$ *is also Hermitian. Now, from Eq.(4), we see that* $\rho_{\mathcal{A}}$ *is a diagonal matrix. So the coefficients* λ_p^2 *are its eigenvalues and the corresponding eigenvectors are* $\{|a_p\rangle\}$. *Similarly,* λ_q^2 *are the eigenvalues and* $\{|b_q\rangle\}$ *are the corresponding eigenvectors of* $\rho_{\mathcal{B}}$. *Hence, from Eq.(2), we see that the Schmidt decomposition of any* $|\Psi\rangle$ *can be found in the following manner:*

$$|\Psi\rangle = \sum_n \sqrt{\lambda_n^2} |a_n\rangle|b_n\rangle. \tag{6}$$

(ii) Does the two qubit state $|\,\Psi\,\rangle = (|\,00\,\rangle + |\,01\,\rangle + |\,10\,\rangle + |\,11\,\rangle)/2$ take the form of Schmidt decomposition? Why/why not? If not, which is a Schmidt decomposition?
Solution: No, the above qubit state is not in a Schmidt decomposition form. If we wanted to push it into that form, we would need to use basis vectors $|\,0\,\rangle$ *and* $|\,1\,\rangle$ *for each qubit* more than once, *and that is not allowed in the Schmidt form. To find out the Schmidt decomposition of this state, we use the prescription that we proved in (i). Here, assuming* $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, *we get,*

$$\rho_{\mathcal{A}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \textit{and } \rho_{\mathcal{B}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The eigenvalues of $\rho_{\mathcal{A}/\mathcal{B}}$ *are* $0, 1$ *and the corresponding eigenvectors are* $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ *and* $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. *Therefore, the Schmidt decomposition of* $|\,\Psi\,\rangle$ *is:*

$$\begin{aligned} |\,\Psi\,\rangle &= 1 \times \frac{1}{\sqrt{2}}(|\,0\,\rangle + |\,1\,\rangle) \otimes \frac{1}{\sqrt{2}}(|\,0\,\rangle + |\,1\,\rangle) + 0 \times \frac{1}{\sqrt{2}}(|\,0\,\rangle - |\,1\,\rangle) \otimes \frac{1}{\sqrt{2}}(|\,0\,\rangle - |\,1\,\rangle) \\ &= \frac{1}{\sqrt{2}}(|\,0\,\rangle + |\,1\,\rangle) \otimes \frac{1}{\sqrt{2}}(|\,0\,\rangle + |\,1\,\rangle). \end{aligned} \tag{7}$$

(which we also could have guessed directly from the input).

(iii) Now consider, as an example for (i), two spin-1/2 particles, labelled A and B, in a (normalized) pure state,

$$|\,\Psi\,\rangle = (|\uparrow\uparrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\downarrow\rangle)/2 \tag{8}$$

Obtain the Schmidt-decomposition by computing and diagonalizing the two reduced density matrices.
Solution: Here,

$$\rho_{\mathcal{A}/\mathcal{B}} = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right).$$

So they are already in the diagonal form with eigenvalues $\frac{1}{2}, \frac{1}{2}$ and corresponding eigenvectors $|\uparrow\rangle$ and $|\downarrow\rangle$. Therefore, the Schmidt decomposition is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right). \tag{9}$$