Phys 637, II-Semester 2020/21, Tutorial 3 solution

We suggest to do "Stages" in the order below, feel free to change that as per your interests. Discuss first on your table within your team, then with neighboring tables.

Stage 1 *(Classical and quantum uncertainties)* Revisit the mixture of coherent states discussed in Q2 of tutorial 2 (please reread the physical discussion of its origin on the tutorial sheet), where

$$\hat{\rho} = \int_0^\infty dr \;\; p(r) | \, \alpha(r) \, \rangle \langle \, \alpha(r) \, |, \tag{1}$$

with $p(r) = \frac{1}{\sqrt{2\pi}\Delta r} e^{-\frac{(r-r_0)^2}{2\Delta r^2}}$.

(i) Find the position uncertainty $\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$ in that density matrix. Discuss the meaning of the formula that you obtain.
Hints: For evaluating quantum expectation values, use the position representation of the coherent state $\langle \, x \, | \, \psi_{coh} \, \rangle = \mathcal{N} e^{-\frac{(x-r)^2}{2\sigma^2}}$ and known facts about the mean (r) and variance (σ^2) of this Gaussian wavefunction. Solution: The density matrix can be written in a simplified version as:

$$\hat{\rho} = \int drp(r) | \, \Psi_{coh}(r) \, \rangle \langle \, \Psi_{coh}(r) \, | \tag{2}$$

with $| \, \Psi_{coh}(r) \, \rangle = \sum_n c_n(r) | \, n \, \rangle$. With the given definitions of density matrix $\hat{\rho}$, we can then write $\langle \hat{x} \rangle$ as:

$$\begin{aligned}
\langle \hat{x} \rangle &= Tr[\hat{\rho}\hat{x}], \\
&= Tr[\int dr \; p(r) | \, \Psi_{coh}(r) \, \rangle \langle \, \Psi_{coh}(r) \, | \hat{x}], \\
&\overset{trace_linear}{=} \int dr \; p(r) \, Tr[| \, \Psi_{coh}(r) \, \rangle \langle \, \Psi_{coh}(r) \, | \hat{x}], \\
&= \int drp(r) \underbrace{\langle \, \Psi_{coh}(r) \, | \hat{x} | \, \Psi_{coh}(r) \, \rangle}_{=r}, \qquad (3) \\
&= \int drp(r)r = r_0. \qquad (4)
\end{aligned}$$

In the fourth equality we have converted the expectation value in the pure state $| \, \Psi_{coh}(r) \, \rangle$ from density matrix notation, to scalar product notation, and then used the fact that the coherent state in position space is symmetric around r, hence needs to have expectation value r. The same result can be used for the last equality again, due to which we know that the mean value of r in the distribution $p(r)$ that was given, is r_0.

Similarly, we can write $\langle \hat{x}^2 \rangle$ as:

$$
\begin{aligned}
\langle \hat{x}^2 \rangle &= Tr[\hat{\rho}\hat{x}^2], \\
&= \int drp(r) \underbrace{\langle\, \Psi_{coh}\, | \hat{x}^2 |\, \Psi_{coh}\, \rangle}_{=\sigma^2+r^2}, &(5) \\
&= \int drp(r)(\sigma^2 + r^2), \\
&= \sigma^2 + \Delta r^2 + r_0^2 &(6)
\end{aligned}
$$

here again we have used the properties of Gaussian integrals a few times. Combining these into Δx we find

$$
\begin{aligned}
\Delta x &= \sqrt{\hat{x}^2 - \hat{x}^2}, \\
&= \sqrt{\sigma^2 + \Delta r^2 + r_0^2 - r_0^2}, \\
&= \sqrt{\sigma^2 + \Delta r^2}. &(7)
\end{aligned}
$$

This illustrates nicely, how the final position uncertainty is a combination of the quantum uncertainty σ in each Gaussian state, and the classical uncertainty Δr due to having a mixture of many different of these Gaussian. If the magnitude of these two is very different, Δx will always be the bigger one.

(ii) (extension outside tutorial) With a computer program, you can plot all density matrix elements ρ_{nm} of $\hat{\rho} = \sum_{nm} \rho_{nm} |\, n\,\rangle\langle\, m\,|$ from Eq. 1 as for example colorful squares in an $N \times N$ array (use e.g. `pcolor.m` matlab). Do that for parameters $\hbar = m = \omega = 1$ (hence $\sigma = 1$) and $r_0 = 10\sigma$, $\Delta r = 4\sigma$ and discuss. You may first have to identify a suitable N to make sure all coherent states are represented correctly. *Solution:*

$$
\begin{aligned}
\hat{\rho} &= \int dr\;\; p(r) \sum_{nm} c_n(r) c_m^*(r) |\, n\,\rangle\langle\, m\,|, \\
&= \sum_{nm} \int dr\;\; p(r) c_n(r) c_m^*(r) |\, n\,\rangle\langle\, m\,|, \\
&= \sum_{nm} \rho_{nm} |\, n\,\rangle\langle\, m\,|, &(8)
\end{aligned}
$$

where $\rho_{nm} = \int dr\;\; p(r) c_n(r) c_m^*(r)$. We could remove the radial dependency by integrating, but that will get messy. Since we are supposed to plot it with a computer anyway, we can instead discretize the integration above by writing:

$$
\hat{\rho} = \sum_{nm} \sum_{k} \;\; p(r_k) c_n(r_k) c_m^*(r_k) |\, n\,\rangle\langle\, m\,|, \tag{9}
$$

for some selected r_k. See `Assignment3_plot_gaussianmixture_v1.m` *online. Using Two different sets of parameters, the results are show in Fig. 1. An extended Gaussian along the digonal can be seen in (1), which is purely a consequence of the incoherent mixing of the coherent states.*

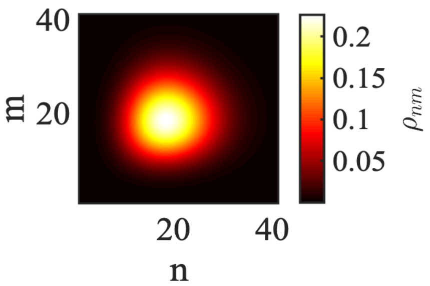

Figure 1: The density matrix elements ρ_{nm} for $\sigma = 1$, $r_0 = 6$ and two values of $\Delta r = 0.2$ (left) and $\Delta r = 3$ (right). On the left we thus almost see an unchanged coherent state. On the right, probabilities ρ_{nn} are smeared out along the diagonal, but mostly *without* coherences between them.