Phys 637,I-Semester 2022/23,Tutorial 2 solution

Stage 1 *(Density matrices)*

(i) Motivation for the density matrix concept: Consider two spins, where the first is the "system", and try to devise a "reduced state of the system only", that has the property that it correctly describes all expectation values of operators on the system. Let us take:

$$|\Psi\rangle = \sqrt{\frac{2}{6}}|\uparrow,\uparrow\rangle + \frac{1}{\sqrt{6}}|\uparrow,\downarrow\rangle + \frac{1}{\sqrt{6}}|\downarrow,\uparrow\rangle + \sqrt{\frac{2}{6}}|\downarrow,\downarrow\rangle,$$
$$\hat{O}_1 = \sigma_x^{(1)}, \tag{1}$$

as a test case *Hint: This should not work in general. In case you find a construction where it does for this example, try it on another one.*
Non-Solution: Will be completed later, ignore for Quiz.

(ii) For the following density matrices of a two-spin system, calculate the reduced density matrix for the first spin only, and then its purity and von-Neumann entropy. We used the basis ordering: $\{|\uparrow,\uparrow\rangle, |\uparrow,\downarrow\rangle, |\downarrow,\uparrow\rangle, |\downarrow,\downarrow\rangle\}$. Discuss.

$$\hat{\rho}_1 = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \qquad \hat{\rho}_2 = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}. \tag{2}$$

Also write the state from (i) as a density matrix.
Solution: We already used the block-structure proposed in solution of assignment one, so we can find the reduced density matrices by tracing each 2 by 2 subblock. Then

$$\hat{\rho}_{red,1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3}$$

Thus

$$\hat{\rho}_{red,1}\hat{\rho}_{red,1} = \frac{1}{4}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \hat{\rho}_{red,1}, \tag{4}$$

hence the trace is one and the purity one. For the von-Neumann entropy we need to diagonalise the matrix, is has eigenvectors $\mathbf{v} = [1, +1]^T/\sqrt{2}$, *with eigenvalue* $+1$ *and* $\mathbf{v} = [1, -1]^T/\sqrt{2}$, *with eigenvalue* 0. *Thus using the formula (3.12) of the lecture* $S(\hat{\rho}_{red,1}) = -\sum_k \lambda_k log_2(\lambda_k) = -(1\log_2 1 + 0\log_2 0) = 0$, *we find the entropy vanishes. Both,* $P = 1$ *and* $S = 0$, *flag a separable state.*
A similar approach for the second matrix gives

$$\hat{\rho}_{red,2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5}$$

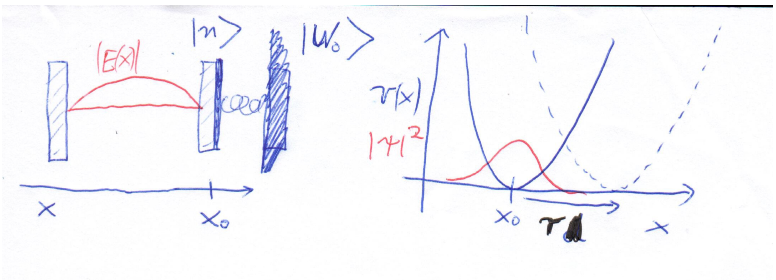
Thus

$$\hat{\rho}_{red,2}\hat{\rho}_{red,2} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

Thus $P = 1/2$, and diagonalisation of $\hat{\rho}_{red,2}$ gives $\mathbf{v} = [1,0]^T$, with eigenvalue $+1/2$ and $\mathbf{v} = [0,1]^T$, with eigenvalue $+1/2$, thus $S(\hat{\rho}_{red,1}) = -\sum_k \lambda_k log_2(\lambda_k) = -(0.5\log_2 0.5 + 0.5\log_2 0.5) = -\log_2 0.5 = +1$.
Here $P < 1$ and $S > 0$ both indicate an entangled state.
Finally, we write the state (1) as a density matrix in the form:

$$\hat{\rho} = \frac{1}{4}\begin{pmatrix} \frac{2}{6} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & \frac{2}{6} \\ \frac{\sqrt{2}}{6} & \frac{1}{6} & \frac{1}{6} & \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{6} & \frac{1}{6} & \frac{1}{6} & \frac{\sqrt{2}}{6} \\ \frac{2}{6} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & \frac{2}{6} \end{pmatrix}. \tag{7}$$

(iii) Pick one of the density matrices in (ii), and change basis to left/right for each spin.
Solution: Will be completed later, ignore for Quiz.



Stage 2

Consider an opto-mechanical quantum harmonic oscillator as shown in the sketch above. Suppose an experiment succeeded to cool that oscillator to the quantum ground-state and then suddenly (approximately instantaneously) displace the minimum of the potential by the distance r as shown. This could be done e.g. by generating a constant radiation pressure force due to light in an optical cavity. This sudden shift creates a coherent state with parameter $\alpha = r/(\sqrt{2}\sigma)$, where $\sigma = \sqrt{\hbar/(m\omega)}$ is the zero point width. First write the density matrix of the oscillator in that coherent state.
Now suppose due to technical noise, the displacement parameter r in the experiment is itself normally distributed with variance $(\Delta r)^2$ and mean r_0. Propose a density matrix in terms of oscillator eigenstates $|\,n\,\rangle$ that describes the experimental situation after the shift, including this imperfection.
Solution: The coherent state is $|\,\alpha\,\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|\,n\,\rangle$, with α linked to r in the question. We can write this as $|\,\alpha\,\rangle = \sum_{n=0}^{\infty} c_n(r)|\,n\,\rangle$ and with this short-hand find $\hat{\rho}(r) = |\,\alpha\,\rangle\langle\,\alpha\,| = \sum_{nm} c_n(r)c_m^*(r)|\,n\,\rangle\langle\,m\,|$, which is a density matrix

for a pure state.
Once we include the technical noise, we have to change this into $\hat{\rho} = \sum_k p(r_k)\hat{\rho}(r_k)$*, where* $p(r_k)$ *is the probability for displacement* r_k*. The question suggested these are distributed according to* $p(r) = \frac{1}{\sqrt{2\pi}\Delta r}\exp\left[-(r-r_0)^2/(2\Delta r^2)\right]$*. If we turn the sum into an integration, we can also write*

$$\hat{\rho} = \int dr\; p(r)\sum_{nm} c_n(r)c_m^*(r)|\, n\,\rangle\langle\, m\,|. \tag{8}$$

Stage 3 *(Von-Neumann measurements)* Consider a simple quantum optics experiment, where you prepared a very short light pulse from a laser beam (that would be in a coherent state wrt. photon number n) and then measure the number of photons with a detector that has single photon resolution. Describe the measurement in terms of the Von-Neumann scheme using steps $\rightarrow$. Define appropriate detector states to write this schematic. Also include the experimenter's brain in the scheme. How does this scheme differ from measurement postulate P3? Which property of Schrödinger's equation is crucial?
Solution: We can write a sequence

$$\begin{aligned} |\,\Psi\,\rangle &= \left(\sum_n c_n|\,n\,\rangle\right)\otimes|\,a_r\,\rangle\otimes|\,b_r\,\rangle \\ &\overset{measure}{\rightarrow}\sum_n c_n\Big(|\,n\,\rangle\otimes|\,a_n\,\rangle\Big)\otimes|\,b_r\,\rangle \\ &\overset{see}{\rightarrow}\sum_n c_n\Big(|\,n\,\rangle\otimes|\,a_n\,\rangle\otimes|\,b_n\,\rangle\Big), \end{aligned} \tag{9}$$

where $|\,a_n\,\rangle$ *means the apparatus has measured photon number* n*, and* $|\,b_n\,\rangle$ *means our brain has seen that the apparatus indicates that it has measured photon number* n*. Crucially this differs from P3 by there being* <u>*no collapse*</u> *of any kind. Instead of a collapse, everything goes into a massively entangled quantum state. All this relies crucially on the* <u>*linearity*</u> *of the TDSE.*

Stage 4 *(Bonus)* For the density matrix in Stage 2 including technical imperfections, find a formula for the position uncertainty. *Solution: See solution of assignment 3 later.*