Phys 637, I-Semester 2022/23, Tutorial 1 solution

Stage 1 *(Course motivation)*

(i) What is an open quantum system? Make a list of some quantum systems you can think of, and for each of them decide whether you have to treat it like an open quantum system or whether you can approximate it as a closed system. Before starting that, think whether besides the type of system, some other information might be important for this classification. Which open questions on "open quantum systems " do you have?
Solution: An open quantum system is a system we have to treat quantum mechanically, but which is in contact with some "outside environment". Regarding the list of systems, of course formally the only truly closed quantum system is the universe. However in practice, whether we treat a system as open or not depends on how strong its interaction with the environment is, together with for how long timescales (or how high energy scales) we are interested in the system. We will see the formal reason for this later in the lecture (please remind me if I forget to point it out in detail). There is of course no perfect solution to this list, the purpose of that question was to get you to discuss. However some possible discussion items would look like this:

- *proton anti-proton colliding in LHC, closed, collision time very short or interaction with any perturbing particle very unlikely.*
- *Atom in excited state in vacuum and for times much shorter than radiative lifetime: closed*
- *Atom in excited state in vacuum for times comparable to radiative lifetime: open, environment is QED vacuum*
- *Atom in excited state in thermal gas environment: open, environment other colliding atoms. Closed if time-scale of interest is much less than collision time.*

Please share your list of "open questions on open quantum systems" with me at this stage, I hope we shall address them throughout the course, if not I might add those pieces.

(ii) What is meant by coherence? Make a list of coherent wave phenomena versus incoherent ones. Which types of coherence do you know. What open questions on coherence do you have?
Solution: Coherence (or not) is a property of ensembles or groups of waves. Coherence implies that across the ensemble or group mutual phase relations are fixed, so that we can make meaningful statements on whether waves will constructively or destructively interfere. Alternating constructive versus destructive interference then often gives rise to interference fringes. Which type of coherence we are talking about then depends on which ensemble or group of waves we look at, and in which domain we look at the interference fringes.

All waves in a group add up to a resultant wave due to the superposition principle. If in that resultant wave, from knowing the phase at a certain point in space, I can infer it at another point in space, we have "spatial coherence". If from knowing it at a certain time, I can infer the phase at another time, I have "temporal coherence". If we Fourier transform the resultant wave and look at Fourier coefficients at different frequencies, if these have a fixed phase relation, we have "spectral coherence".
Examples of coherent waves: Laser light, pattern behind double slit for incoming plane-wave. Examples of incoherent waves: Light form a light bulb, pattern behind double slit if distance between slits is larger than the coherence length.

Stage 2 *(Quantum mechanics)*

(i) How do we construct quantum mechanics (on which formal basis)? List some successes of quantum mechanics. What shortcomings of quantum mechanics do you know?
Solution: We can derive QM from 5 postulates (section 1.5.1). These cannot themselves be derived from something else. QM so far has passed every test that we know, some to spectacular precision. For example the anomalous magnetic moment of the electron (electron g-factor), is in agreement between experiment and QED calculations up to 10 signifiant digits. Few shortcomings are that we have not succeeded yet to quantise gravity and that a measurement apparatus has to be dealt with using postulate P3 rather than using the TDSE.

(ii) How do we go from single-particle quantum mechanics to many-particle quantum mechanics? Why does it typically get nasty?
Solution: The wavefunctions gets on extra coordinate (index) for each added particle with a continuous (discrete) degree of freedom. This gets nasty because we end up with a very high dimensional wavefunction, that is hard to visualize, hard to do analytical math with, and hard to store or process in a computer

(iii) Write the following quantum states (some many-body) as an equation:

- One electron (3D position $\mathbf{r}_1$) is near a proton in the ground-state of the Hydrogen atom, and a second electron is free and has a well defined momentum $\mathbf{p}_2$.
Solution:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \varphi_{100}(\mathbf{r}_1)e^{i\mathbf{p}_2\cdot\mathbf{r}_2/\hbar}, \tag{1}$$

where $\varphi_{100}(\mathbf{r}_1)$ is the electronic ground-state wave function of the Hydrogen atom.

- An electron moving in one dimension travels to the left (momentum $-p$) if its spin projection along the x-axis is $m_s = +1/2$ and to the right (momentum $+p$) otherwise. Travelling to the right is twice as likely as to the left. *Solution:*

$$\langle\, x \,|\, \Psi \,\rangle = \sqrt{1/3}e^{-ikx}|\!\uparrow\rangle + \sqrt{2/3}e^{ikx}|\!\downarrow\rangle. \tag{2}$$

- One harmonic oscillator is in an arbitrary state, and a second harmonic oscillator is in the exact same quantum state. *Solution:*

$$|\,\Psi\,\rangle = (\sum_n c_n|\,n\,\rangle) \otimes (\sum_m c_m|\,m\,\rangle). \tag{3}$$

- One harmonic oscillator is in an arbitrary state, but if you measure the energy of a second harmonic oscillator it always is found to be the same as the energy of the first. *Solution:*

$$|\,\Psi\,\rangle = \sum_n c_n|\,n\,\rangle \otimes |\,n\,\rangle. \tag{4}$$

- Which of the above states are entangled?
  Solution: Eq. (1) and Eq. (3) are clearly separable (see lecture), hence they are not entangled. In Eq. (2) we have only one particle, but two degrees of freedom DGFs (spin and motion). You cannot write this as a product of some wavefunction for motion and some for spin, hence these two DGFs are entangled. As long as at least two different c_n are nonzero in Eq. (4), that can also not be written as a product and hence is entangled.

(iv) Write the following Hamiltonian of two interacting particles (different masses m and M) in 1D in terms of their ladder operators (defining these explicitly in terms of quantities given).

$$\hat{H} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial r_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial r_2^2} + \frac{1}{2}M\omega_1^2 r_1^2 + \kappa r_2^2 + \eta(r_1 - r_2)^2. \tag{5}$$

Solution: We use the usual definitions of ladder operators, but we have to take care that each oscillator has different masses and frequencies. To find the latter, let us first rewrite the prefactor of r_2^2 as $\kappa = m\omega_2^2/2$, hence $\omega_2 = \sqrt{2\kappa/m}$. (the original sheet had the wrong mass written in the prefactor of r_1^2, taking that serious one would have to do a similar re-definition of frequency there as well). Then

$$\begin{aligned}
\hat{r}_1 &= \sqrt{\frac{\hbar}{2M\omega_1}}(\hat{b}_1 + \hat{b}_1^\dagger), \\
\hat{p}_1^\dagger &= i\sqrt{2M\omega_1\hbar}\left(\hat{b}_1^\dagger - \hat{b}_1\right), \\
\hat{r}_2 &= \sqrt{\frac{\hbar}{2\sqrt{2m\kappa}}}(\hat{b}_2 + \hat{b}_2^\dagger), \\
\hat{p}_2^\dagger &= i\sqrt{2\sqrt{2m\kappa}\hbar}\left(\hat{b}_2^\dagger - \hat{b}_2\right),
\end{aligned} \tag{6}$$

using the inversion of Eq. (1.23) [i.e. below Eq. (2.6)]. We know that the non-interacting part of (5) will just be $\hbar\omega_k(\hat{b}_k^\dagger\hat{b}_k + 1/2)$ for each oscillator, hence we only have to substitute (6) into the interaction term $\sim \eta$ to find:

$$\hat{H} = \hbar\omega_1(\hat{b}_1^\dagger\hat{b}_1 + 1/2) + \hbar\sqrt{2\kappa/m}(\hat{b}_2^\dagger\hat{b}_2 + 1/2) + \left(\sqrt{\frac{\hbar}{2M\omega_1}}(\hat{b}_1 + \hat{b}_1^\dagger) - \sqrt{\frac{\hbar}{2\sqrt{2m\kappa}}}(\hat{b}_2 + \hat{b}_2^\dagger)\right)^2 . \quad (7)$$

(v) Interpret the following pieces of a many-body Hamiltonian (interactions) as a sentence that describes the physics of the interaction in question (Pauli matrices refer to a spin 1/2 object, ladder operators to another object or the motional degree of freedom of the same object, treated as a harmonic oscillator).

- $H_{\text{int}} = \hat{\sigma}_z(\hat{a} + \hat{a}^\dagger)$.
  Solution: The spin's energy depends on position $\sim x$ for $\uparrow$ and $\sim -x$ for $\downarrow$
- $H_{\text{int}} = 2\hbar\omega|\uparrow\rangle\langle\uparrow|\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega|\downarrow\rangle\langle\downarrow|\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$.
  Solution: The frequency of some harmonic oscillator depends on the spin.
- $H_{\text{int}} = \kappa\left(|\uparrow\rangle\langle\downarrow|\hat{a} + |\downarrow\rangle\langle\uparrow|\hat{a}^\dagger\right)$.
  Solution: An oscillator quantum can be absorbed while raising the spin, and the reverse.

Stage 3 *(Markovianity)* Come up with some further examples of Markovian and non-Markovian stochastic processes, beyond the two given as example in the lecture notes.
No-solution: I am sure you can find many examples in the internet.