Phys 637, I-Semester 2022/23, Assignment 6

Instructor: Sebastian Wüster
Due-date: 11.11.2022

(1) Two level atom
First part: Thermal equilibrium with a radiation field: Assume a two-level atom in a radiation field at temperature T as in example 37, section 4.7, described by the optical Bloch equations (4.89). First, we *do NOT* assume any additional laser coupling ($\Omega = 0$, $\Delta = 0$)
(1a) Find the steady state of the atom $\hat{\rho}^{(ss)}$ at temperature T. Use the expression for the thermal occupation of the resonant photon mode $N_{\omega_{eg}}(T)$ from the lecture. [4 points]

(1b) Determine the ratio of steady state atomic population in the excited and ground state and discuss your result. [2 points]

Second part: Steady state in a laser drive:
(1c) Now move to zero temperature ($T = 0$, $N_{\omega_{eg}}(T) = 0$), but assume the presence of coherent coupling $\Omega \neq 0$, $\Delta \neq 0$). Find the steady state under these conditions. Compare with the figures in example 37, section 4.7 of the lecture and discuss. [2 points]

(2) Wigner function evolution equation
Using the relation

$$W(x,p,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\ e^{ipy}\ \rho(x - \frac{y}{2}, x + \frac{y}{2}, t) \tag{1}$$

it is possible to turn the evolution equation for the position-space representation of the density matrix, $\rho(x, x', t)$ in Eq. (4.57) into one for the Wigner function $W(x, p, t)$. Show that the result is a Fokker-Planck type equation for the Wigner function

$$\begin{aligned}\frac{\partial}{\partial t}W(x,p,t) = \Bigg[&- \frac{P}{M}\frac{\partial}{\partial x} + M(\Omega^2 + \tilde{\Omega}^2)x\frac{\partial}{\partial p} + \gamma\frac{\partial}{\partial p}p \\ &+ D\frac{\partial^2}{\partial p^2} - f\frac{\partial}{\partial x}\frac{\partial}{\partial p}\Bigg]W(x,p,t).\end{aligned} \tag{2}$$

The name "Fokker-Planck equation" comes from statistical mechanics where it describes some evolution equations of probability distributions.
Hints: (i) You have to use integrations by parts together with $\rho(x, x') \to 0$ at $x = \pm\infty$ or $x' = \pm\infty$. (ii) Also note that $ye^{ipy} = -i(\partial/\partial p)e^{ipy}$. (iii) You may need $\frac{\partial \rho}{\partial f} = \frac{1}{2}\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial y}$ and $\frac{\partial \rho}{\partial g} = \frac{1}{2}\frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial y}$, where $f = x + y/2$ and $f = x - y/2$. You can show this using the usual transformation rules for multi-variate derivatives. [6 points]

(3) Quantum Brownian motion numerically

Now let us solve the equation we found in Q2 numerically.

(3a) Show that a derivative amounts to multiplication in Fourier space:

$$\frac{\partial}{\partial x} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \left[\underbrace{(ik)}_{\to \text{XMDS}} \quad \tilde{f}(k) \right], \tag{3}$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$ and we use the symmetric $\frac{1}{\sqrt{2\pi}}$ convention. Then find the corresponding Fourier-space expressions for all other derivatives that occur in the equation of part (2). Do NOT treat x and p as Fourier pair, instead each coordinate get's their own Fourier coordinate: $x \leftrightarrow k_x$, $p \leftrightarrow k_p$, this is because we want to evolve a 2D function $W(x, p)$ and each dimension has be treated separately. [3 points]

(3b) Insert your derivatives at $XXXX$ in `Assignment6_program_draft_v4.xmds`. For each derivative operator you only write the part marked with $\underbrace{\quad}$ in (3). XMDS automatically Fourier transforms the function the operator acts on, multiplies with e.g. ik_x and Fourier transforms back to insert into the equation of motion. As a first step, show the solution agrees with Example 35 of the lecture if the same initial state and parameters are chosen (all pre-set). Then play with initial state and parameters to explore the functioning of all the terms (except the f term, which is numerically unstable) in the equation of motion (together and separately). Make plots and discuss. Avoid too large or too small values for parameters. Revisit your guesses in tutorial7, Stage2 and check if they were correct. You may use `Assignment6_wigner_slideshow_v1.m` for visualizing the evolution of the Wigner function. [5 points]