Phys 637, I-Semester 2022/23, Assignment 6 solution

(1) Two level atom

First part: Thermal equilibrium with a radiation field: Assume a two-level atom in a radiation field at temperature T as in example 37, section 4.7, described by the optical Bloch equations (4.89). First, we *do NOT* assume any additional laser coupling ($\Omega = 0$, $\Delta = 0$)

(1a) Find the steady state of the atom $\hat{\rho}^{(ss)}$ at temperature T. Use the expression for the thermal occupation of the resonant photon mode $N_{\omega_{eg}}(T)$ from the lecture. [4 points]

Set the LHS and term in first square brackets on the RHS (containing $\Omega = 0$, $\Delta = 0$) of equation (4.89) to zero. Solve the resultant algebraic system of equations, together with $\rho_{gg} + \rho_{ee} = 1$, e.g. using mathematica, and find:

$$\{\rho_{gg} = e^{\beta\hbar\omega}\rho_{ee}, \rho_{eg} = 0, \rho_{ge} = 0\}$$

(1b) Determine the ratio of steady state atomic population in the excited and ground state and discuss your result. [2 points]

Based on the above, the ratio is

$$\frac{\rho_{ee}}{\rho_{gg}} = e^{-\beta\hbar\omega}.$$

As $T \to 0$, $\rho_{ee} \to 0$, and ρ_{ee} increases with T. The interpretation is, that the atom now also is in thermal equilibrium with radiation field, at the same temperature T. Because of this the ratio of occupation probabilities contains the Boltzmann factor $e^{-\beta\hbar\omega} = e^{-\Delta E/k_B T}$, where ΔE is the energy difference between ground- and excited state.

Second part: Steady state in a laser drive:

(1c) Now move to zero temperature ($T = 0$, $N_{\omega_{eg}}(T) = 0$), but assume the presence of coherent coupling $\Omega \neq 0$, $\Delta \neq 0$). Find the steady state under these conditions. Compare with the figures in example 37, section 4.7 of the lecture and discuss. [2 points]

Set the LHS of equation (4.89) to zero, on the RHS use that $N_{\omega_{eg}}(T = 0) = 0$ to significantly simplify. Solve the resultant algebraic system of equations, together with $\rho_{gg} + \rho_{ee} = 1$, e.g. using mathematica, and find:

$$\rho_{gg} = \frac{\Omega^2 + \gamma^2 + 4\Delta^2}{2\Omega^2 + \gamma^2 + 4\Delta^2}, \quad \rho_{ge} = \frac{i\Omega\gamma + 2\Delta\Omega}{2\Omega^2 + \gamma^2 + 4\Delta^2},$$

$$\rho_{eg} = \frac{-i\Omega\gamma + 2\Delta\Omega}{2\Omega^2 + \gamma^2 + 4\Delta^2}, \quad \rho_{ee} = \frac{\Omega^2}{2\Omega^2 + \gamma^2 + 4\Delta^2}.$$

In the example we had used parameters $\Delta = 0$, $\Omega/(2\pi) = 1$ and $\gamma/(2\pi) = 4$. Hence

$$\rho_{ee} = \frac{\Omega^2}{2\Omega^2 + \gamma^2 + 4\Delta^2} = \left(\frac{1}{5}\right)^2 = 1/25,$$

which is consistent with the figure.

(2) Wigner function evolution equation
Using the relation

$$W(x,p,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\ e^{ipy}\ \rho(x-\frac{y}{2}, x+\frac{y}{2}, t) \tag{1}$$

it is possible to turn the evolution equation for the position-space representation of the density matrix, $\rho(x,x',t)$ in Eq. (4.57) into one for the Wigner function $W(x,p,t)$. Show that the result is a Fokker-Planck type equation for the Wigner function

$$\frac{\partial}{\partial t}W(x,p,t) = \Bigg[-\frac{P}{M}\frac{\partial}{\partial x} + M(\Omega^2+\tilde{\Omega}^2)x\frac{\partial}{\partial p} + \gamma\frac{\partial}{\partial p}p$$
$$+ D\frac{\partial^2}{\partial p^2} - f\frac{\partial}{\partial x}\frac{\partial}{\partial p}\Bigg]W(x,p,t). \tag{2}$$

The name "Fokker-Planck equation" comes from statistical mechanics where it describes some evolution equations of probability distributions.
Hints: (i) You have to use integrations by parts together with $\rho(x,x') \to 0$ at $x = \pm\infty$ or $x' = \pm\infty$. (ii) Also note that $ye^{ipy} = -i(\partial/\partial p)e^{ipy}$. (iii) You may need $\frac{\partial\rho}{\partial f} = \frac{1}{2}\frac{\partial\rho}{\partial x} + \frac{\partial\rho}{\partial y}$ and $\frac{\partial\rho}{\partial g} = \frac{1}{2}\frac{\partial\rho}{\partial x} - \frac{\partial\rho}{\partial y}$, where $f = x + y/2$ and $f = x - y/2$. You can show this using the usual transformation rules for multi-variate derivatives. [6 points]

Solution: We start from (1) and take the time derivative

$$\frac{\partial}{\partial t}W(x,p,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\ e^{ipy}\ \frac{\partial}{\partial t}\rho(x-\frac{y}{2}, x+\frac{y}{2}, t), \tag{3}$$

hence we require $\frac{\partial}{\partial t}\rho(x-\frac{y}{2}, x+\frac{y}{2}, t)$. Let us first recall Eq. (4.57) from the lecture notes:

$$\frac{\partial}{\partial t}\rho(X,X',t) = \Bigg\{\frac{-i}{2M}\left(\frac{\partial^2}{\partial X'^2} - \frac{\partial^2}{\partial X^2}\right) - \frac{i}{2}M(\Omega^2+\tilde{\Omega}^2)(X^2 - X'^2)$$
$$+ \gamma(X-X')\left(\frac{\partial}{\partial X'} - \frac{\partial}{\partial X}\right) - D(X-X')^2 +$$
$$iF(X-X')\left(\frac{\partial}{\partial X'} + \frac{\partial}{\partial X}\right)\Bigg\}\rho(X,X',t), \tag{4}$$

where we have changed a coefficient name to "F" instead of "f" to avoid the confusion between the constant coefficient and the function defined in the hint. Since we need information on $\rho(x-y/2, x+y/2, t)$, we define $f = x+y/2, g = x-y/2$ as given in the hint. The Eq. (4) can then be re-written as:

$$\frac{\partial}{\partial t}\rho(g,f,t) = \Bigg\{\frac{-i}{2M}\left(\frac{\partial^2}{\partial f^2} - \frac{\partial^2}{\partial g^2}\right) - \frac{i}{2}M(\Omega^2+\tilde{\Omega}^2)(g^2 - f^2)$$
$$+ \gamma(g-f)\left(\frac{\partial}{\partial f} - \frac{\partial}{\partial g}\right) - D(g-f)^2 +$$
$$iF(g-f)\left(\frac{\partial}{\partial f} + \frac{\partial}{\partial g}\right)\Bigg\}\rho(g,f,t). \tag{5}$$

To convert partial derivatives into ones for x and y, we use the hint such that:

$$\frac{\partial \rho}{\partial f} = \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial f} + \frac{\partial \rho}{\partial y}\frac{\partial y}{\partial f} = \frac{1}{2}\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial y}, \tag{6}$$

$$\frac{\partial \rho}{\partial g} = \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial g} + \frac{\partial \rho}{\partial y}\frac{\partial y}{\partial g} = \frac{1}{2}\frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial y}. \tag{7}$$

For this we also needed $x = (f+g)/2$ and $y = f - g$. Second derivatives then follow as

$$\frac{\partial^2 \rho}{\partial f^2} = \frac{1}{4}\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial x \partial y} + \frac{\partial^2 \rho}{\partial y^2} \tag{8}$$

$$\frac{\partial^2 \rho}{\partial g^2} = \frac{1}{4}\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial^2 \rho}{\partial x \partial y} + \frac{\partial^2 \rho}{\partial y^2}, \tag{9}$$

Using these in Eq. (5) yields:

$$\frac{\partial}{\partial t}\rho(g,f,t) = \Bigg\{\frac{-i}{M}\left(\frac{\partial^2}{\partial x \partial y}\right) + iM(\Omega^2 + \tilde{\Omega}^2)(xy)$$

$$+ \gamma(y)\left(2\frac{\partial}{\partial y}\right) - Dy^2 + iF(y)\left(\frac{\partial}{\partial x}\right)\Bigg\}\rho(g,f,t). \tag{10}$$

Insertion into Eq. (3) gives

$$\frac{\partial}{\partial t}W(x,p,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\ e^{ipy}\ \Bigg\{\frac{-i}{M}\left(\frac{\partial^2}{\partial x \partial y}\right) + iM(\Omega^2 + \tilde{\Omega}^2)(xy)$$

$$+ \gamma(y)\left(2\frac{\partial}{\partial y}\right) - Dy^2 + iF(y)\left(\frac{\partial}{\partial x}\right)\Bigg\}\rho(x - \frac{y}{2}, x + \frac{y}{2}, t). \tag{11}$$

We can now convert $\frac{\partial}{\partial y}$ into p and y into $\frac{\partial}{\partial p}$ using integration by parts, as per hints provided in the question. We show here for example the integration by part for the first term:

$$\frac{-i}{M}\int_{-\infty}^{\infty} dy\ e^{ipy}\ \frac{\partial^2}{\partial x \partial y}\rho(x - \frac{y}{2}, x + \frac{y}{2}, t) \tag{12}$$

$$= \frac{-i}{M}\Bigg\{\underbrace{e^{ipy}\frac{\partial}{\partial x}\rho(x - \frac{y}{2}, x + \frac{y}{2}, t)\Bigg|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} dy\ \left(\frac{\partial}{\partial y}e^{ipy}\right)\frac{\partial}{\partial x}\rho(x - \frac{y}{2}, x + \frac{y}{2}, t)\Bigg\}$$

$$= -ip\frac{\partial}{\partial x}\underbrace{\int_{-\infty}^{\infty} dy\ e^{ipy}\ \rho(x - \frac{y}{2}, x + \frac{y}{2}, t)}_{=W(x,p,t)}\Bigg\}$$

$$= -\frac{p}{M}\frac{\partial}{\partial x}W(x,p,t). \tag{13}$$

After similar tricks for all other terms, we reach Eq. (2).

(3) Quantum Brownian motion numerically
Now let us solve the equation we found in Q2 numerically.
(3a) Show that a derivative amounts to multiplication in Fourier space:

$$\frac{\partial}{\partial x} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \left[\underbrace{(ik)}_{\to \text{XMDS}} \tilde{f}(k) \right], \tag{14}$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$ and we use the symmetric $\frac{1}{\sqrt{2\pi}}$ convention. Then find the corresponding Fourier-space expressions for all other derivatives that occur in the equation of part (3). Do NOT treat x and p as Fourier pair, instead each coordinate get's their own Fourier coordinate: $x \leftrightarrow k_x$, $p \leftrightarrow k_p$, this is because we want to evolve a 2D function $W(x, p)$ and each dimension has be treated separately. [3 points]

Solution: We know that $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \tilde{f}(k)$. *When we take the derivative* $\frac{\partial}{\partial x} f(x)$ *we can interchange the order of differentiation and integration to find:*

$$\frac{\partial}{\partial x} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left[\frac{\partial}{\partial x} e^{ikx} \right] \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \left[(ik) e^{ikx} \right] \tilde{f}(k) \tag{15}$$

(3b) Insert your derivatives at $XXXX$ in `Assignment6_program_draft_v4.xmds`. For each derivative operator you only write the part marked with $\underbrace{\quad}$ in (15). XMDS automatically Fourier transforms the function the operator acts on, multiplies with e.g. ik_x and Fourier transforms back to insert into the equation of motion. As a first step, show the solution agrees with Example 35 of the lecture if the same initial state and parameters are chosen (all pre-set). Then play with initial state and parameters to explore the functioning of all the terms (except the f term, which is numerically unstable) in the equation of motion (together and separately). Make plots and discuss. Avoid too large or too small values for parameters. Revisit your guesses in tutorial7, Stage2 and check if they were correct. You may use `Assignment6_wigner_slideshow_v1.m` for visualizing the evolution of the Wigner function. [5 points] *See solution of tutorial 7*